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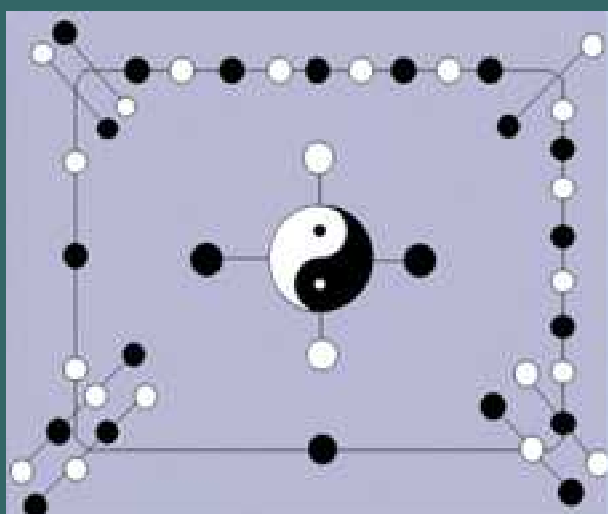
Edited By Linfan MAO



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By A. Einstein, an American theoretical physicist.

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Neutrosophic Rings I

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Abstract: In this paper, we present some elementary properties of neutrosophic rings. The structure of neutrosophic polynomial rings is also presented. We provide answers to the questions raised by Vasantha Kandasamy and Florentin Smarandache in [1] concerning principal ideals, prime ideals, factorization and Unique Factorization Domain in neutrosophic polynomial rings.

Key Words: Neutrosophy, neutrosophic, neutrosophic logic, fuzzy logic, neutrosophic ring, neutrosophic polynomial ring, neutrosophic ideal, pseudo neutrosophic ideal, neutrosophic R-module.

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§1. Introduction

Neutrosophy is a branch of philosophy introduced by Florentin Smarandache in 1980. It is the basis of neutrosophic logic, neutrosophic probability, neutrosophic set and neutrosophic statistics. While neutrosophic set generalizes the fuzzy set, neutrosophic probability generalizes the classical and imprecise probabilty, neutrosophic statistics generalizes classical and imprecise statistics, neutrosophic logic however generalizes fuzzy logic, intuitionistic logic, Boolean logic, multi-valued logic, paraconsistent logic and dialetheism. In the neutrosophic logic, each proposition is estimated to have the percentage of truth in a subset T, the percentage of indeterminancy in a subset I, and the percentage of falsity in a subset F. The use of neutrosophic theory becomes inevitable when a situation involving indeterminancy is to be modeled since fuzzy set theory is limited to modeling a situation involving uncertainty.

The introduction of neutrosophic theory has led to the establishment of the concept of neutrosophic algebraic structures. Vasantha Kandasamy and Florentin Smarandache for the first time introduced the concept of neutrosophic algebraic structures in [2] which has caused a paradigm shift in the study of algebraic structures. Some of the neutrosophic algebraic structures introduced and studied in [2] include neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, neutrosophic bigroupoids and so on. The study of neutrosophic rings was

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introduced for the first time by Vasantha Kandasamy and Florentin Smarandache in [1]. Some of the neutrosophic rings studied in [1] include neutrosophic polynomial rings, neutrosophic matrix rings, neutrosophic direct product rings, neutrosophic integral domains, neutrosophic unique factorization domains, neutrosophic division rings, neutrosophic integral quaternions, neutrosophic rings of real quarternions, neutrosophic group rings and neutrosophic semigroup rings.

In Section 2 of this paper, we present elementary properties of neutrosophic rings. Section 3 is devoted to the study of structure of neutrosophic polynomial rings and we present algebraic operations on neutrosophic polynomials. In section 4, we present factorization in neutrosophic polynomial rings. We show that Division Algorithm is generally not true for neutrosophic polynomial rings. We show that a neutrosophic polynomial ring $\langle R \cup I \rangle [x]$ cannot be an Integral Domain even if R is an Integral Domain and also we show that $\langle R \cup I \rangle [x]$ cannot be a Unique Factorization Domain even if R is a Unique Factorization Domain. In section 5 of this paper, we present neutrosophic ideals in neutrosophic polynomial rings and we show that every non-zero neutrosophic principal ideal is not a neutrosophic prime ideal.

§2. **Elementary Properties of Neutrosophic Rings**

In this section we state for emphasis some basic definitions and results but for further details about neutrosophic rings, the reader should see [1].

Definition 2.1([1]) *Let $(R, +, .)$ be any ring. The set*

$$\langle R \cup I \rangle = \{a + bI : a, b \in R\}$$

is called a neutrosophic ring generated by R and I under the operations of R.

Example 2.2 $\langle \mathcal{Z} \cup I \rangle$, $\langle \mathcal{Q} \cup I \rangle$, $\langle \mathcal{R} \cup I \rangle$ and $\langle \mathcal{C} \cup I \rangle$ are neutrosophic rings of integer, rational, real and complex numbers respectively.

Theorem 2.3 *Every neutrosophic ring is a ring and every neutrosophic ring contains a proper subset which is just a ring.*

Definition 2.4 *Let $\langle R \cup I \rangle$ be a neutrosophic ring. $\langle R \cup I \rangle$ is said to be commutative if $\forall\ x, y \in \langle R \cup I \rangle$, $xy = yx$.*

If in addition there exists $1 \in \langle R \cup I \rangle$ such that $1.r = r.1 = r$ for all $r \in \langle R \cup I \rangle$ then we call $\langle R \cup I \rangle$ a commutative neutrosophic ring with unity.

Definition 2.5 *Let $\langle R \cup I \rangle$ be a neutrosophic ring. A proper subset P of $\langle R \cup I \rangle$ is said to be a neutrosophic subring of $\langle R \cup I \rangle$ if $P = \langle S \cup nI \rangle$ where S is a subring of R and n an integer. P is said to be generated by S and nI under the operations of R.*

Definition 2.6 *Let $\langle R \cup I \rangle$ be a neotrosophic ring and let P be a proper subset of $\langle R \cup I \rangle$ which is just a ring. Then P is called a subring.*

Definition 2.7 *Let T be a non-empty set together with two binary operations $+$ and $.$ T is said to be a pseudo neutrosophic ring if the following conditions hold:*
(i) *T contains elements of the form $(a+bI)$, where a and b are real numbers and $b \neq 0$ for at least one value;*

(ii) *$(T,+)$ is an Abelian group;*

(iii) *$(T,.)$ is a semigroup;*

(iv) *$\forall\ x, y, z \in T$, $x(y+z) = xy + xz$ and $(y+z)x = yx + zx$.*

Definition 2.8 *Let $\langle R \cup I \rangle$ be any neutrosophic ring. A non-empty subset P of $\langle R \cup I \rangle$ is said to be a neutrosophic ideal of $\langle R \cup I \rangle$ if the following conditions hold:*

(i) *P is a neutrosophic subring of $\langle R \cup I \rangle$;*

(ii) *for every $p \in P$ and $r \in \langle R \cup I \rangle$, $rp \in P$ and $pr \in P$.*

If only $rp \in P$, we call P a left neutrosophic ideal and if only $pr \in P$, we call P a right neutrosophic ideal. When $\langle R \cup I \rangle$ is commutative, there is no distinction between rp and pr and therefore P is called a left and right neutrosophic ideal or simply a neutrosophic ideal.

Definition 2.9 *Let $\langle R \cup I \rangle$ be a neutrosophic ring and let P be a pseudo neutrosophic subring of $\langle R \cup I \rangle$. P is said to be a pseudo neutrosophic ideal of $\langle R \cup I \rangle$ if $\forall\ p \in P$ and $r \in \langle R \cup I \rangle$, $rp, pr \in P$.*

Theorem 2.10([1]) *Let $\langle \mathcal{Z} \cup I \rangle$ be a neutrosophic ring. Then $\langle \mathcal{Z} \cup I \rangle$ has a pseudo ideal P such that*

$$\langle \mathcal{Z} \cup I \rangle \cong \mathcal{Z}_n.$$

Definition 2.11 *Let $\langle R \cup I \rangle$ be a neutrosophic ring.*

(i) *$\langle R \cup I \rangle$ is said to be of characteristic zero if $\forall\ x \in R$, $nx = 0$ implies that $n = 0$ for an integer n;*

(ii) *$\langle R \cup I \rangle$ is said to be of characteristic n if $\forall x \in R$, $nx = 0$ for an integer n.*

Definition 2.12 *An element x in a neutrosophic ring $\langle R \cup I \rangle$ is called a left zero divisor if there exists a nonzero element $y \in \langle R \cup I \rangle$ such that $xy = 0$.*

A right zero divisor can be defined similarly. If an element $x \in \langle R \cup I \rangle$ is both a left and a right zero divisor, it is then called a zero divisor.

Definition 2.13 *Let $\langle R \cup I \rangle$ be a neutrosophic ring. $\langle R \cup I \rangle$ is called a neutrosophic integral domain if $\langle R \cup I \rangle$ is commutative with no zero divisors.*

Definition 2.14 *Let $\langle R \cup I \rangle$ be a neutrosophic ring. $\langle R \cup I \rangle$ is called a neutrosophic division ring if $\langle R \cup I \rangle$ is non-commutative and has no zero divisors.*

Definition 2.15 *An element x in a neutrosophic ring $\langle R \cup I \rangle$ is called an idempotent element if $x^2 = x$.*

Example 2.16 In the neutrosophic ring $\langle \mathcal{Z}_2 \cup I \rangle$, 0 and 1 are idempotent elements.

Definition 2.17 *An element $x = a+bI$ in a neutrosophic ring $\langle R \cup I \rangle$ is called a neutrosophic idempotent element if $b \neq 0$ and $x^2 = x$.*

Example 2.18 In the neutrosophic ring $\langle \mathcal{Z}_3 \cup I \rangle$, I and 1+2I are neutrosophic idempotent elements.

Definition 2.19 *Let $\langle R \cup I \rangle$ be a neutrosophic ring. An element $x = a + bI$ with $a \neq \pm b$ is said to be a neutrosophic zero divisor if there exists $y = c + dI$ in $\langle R \cup I \rangle$ with $c \neq \pm d$ such that $xy = yx = 0$.*

Definition 2.20 *Let $x = a + bI$ with $a, b \neq 0$ be a neutrosophic element in the neutrosophic ring $\langle R \cup I \rangle$. If there exists an element $y \in R$ such that $xy = yx = 0$, then y is called a semi neutrosophic zero divisor.*

Definition 2.21 *An element $x = a+bI$ with $b \neq 0$ in a neutrosophic ring $\langle R \cup I \rangle$ is said to be a neutrosophic nilpotent element if there exists a positive integer n such that $x^n = 0$.*

Example 2.22 In the neutrosophic ring $\langle \mathcal{Z}_4 \cup I \rangle$ of integers modulo 4, 2+2I is a neutrosophic nilpotent element.

Example 2.23 Let $\langle M_{2\times 2} \cup I \rangle$ be a neutrosophic ring of all 2×2 matrices. An element $A = \begin{bmatrix} 0 & 2I \\ 0 & 0 \end{bmatrix}$ is neutrosophic nilpotent since $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Definition 2.24 *Let Let r be a fixed element of the neutrosophic ring $\langle R \cup I \rangle$. We call the set*

$$N(r) = \{x \in \langle R \cup I \rangle : xr = rx\}$$

the normalizer of r in $\langle R \cup I \rangle$.

Example 2.25 Let M be a neutrosophic ring defined by

$$M = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \langle \mathcal{Z}_2 \cup I \rangle \right\}.$$

It is clear that M has 16 elements.

(i) The normalizer of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ in M is obtained as

$$N\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1+I \\ 0 & 0 \end{bmatrix} \right\}.$$

(ii) The normalizer of $\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$ in M is obtained as

$$N\left(\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \right) =$$

$$\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1+I \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1+I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1+I & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1+I & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1+I & 1+I \\ 0 & 0 \end{bmatrix} \right\}.$$

It is clear that $N\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$ and $N\left(\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}\right)$ are pseudo neutrosophic subrings of M and in fact they are pseudo neutrosophic ideals of M. These emerging facts are put together in the next proposition.

Proposition 2.26 *Let N(r) be a normalizer of an element in a neutrosophic ring* $\langle R \cup I \rangle$*. Then*

(*i*) $N(r)$ *is a pseudo neutrosophic subring of* $\langle R \cup I \rangle$*;*

(*ii*) $N(r)$ *is a pseudo neutrosophic ideal of* $\langle R \cup I \rangle$*.*

Definition 2.27 *Let P be a proper subset of the neutrosophic ring* $\langle R \cup I \rangle$*. The set*

$$Ann_l(P) = \{x \in \langle R \cup I \rangle : xp = 0 \ \ \forall \ p \in P\}$$

is called a left annihilator of P and the set

$$Ann_r(P) = \{y \in \langle R \cup I \rangle : py = 0 \ \ \forall \ p \in P\}$$

is called a right annihilator of P. If $\langle R \cup I \rangle$ *is commutative, there is no distinction between left and right annihilators of P and we write Ann(P).*

Example 2.28 Let M be the neutrosophic ring of Example 2.25. If we take

$$P = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1+I & 1+I \\ 0 & 0 \end{bmatrix} \right\},$$

then, the left annihilator of P is obtained as

$$Ann_l(P) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1+I \\ 0 & 0 \end{bmatrix} \right\}$$

which is a left pseudo neutrosophic ideal of M.

Proposition 2.29 *Let* $\langle R \cup I \rangle$ *be a neutrosophic ring and let P be a proper subset of* $\langle R \cup I \rangle$*. Then the left(right) annihilator of P is a left(right) pseudo neutrosophic ideal of* $\langle R \cup I \rangle$*.*

Example 2.30 Consider $\langle \mathcal{Z}_2 \cup I \rangle = \{0, 1, I, 1+I\}$ the neutrosophic ring of integers modulo 2. If $P = \{0, 1+I\}$, then $Ann(P) = \{0, I\}$.

Example 2.31 Consider $\langle \mathcal{Z}_3 \cup I \rangle = \{0, 1, I, 2I, 1+I, 1+2I, 2+I, 2+2I\}$ the neutrosophic ring of integers modulo 3. If $P = \{0, I, 2I\}$, then $Ann(P) = \{0, 1+2I, 2+I\}$ which is a pseudo nuetrosophic subring and indeed a pseudo neutrosophic ideal.

Proposition 2.32 *Let $\langle R \cup I \rangle$ be a commutative neutrosophic ring and let P be a proper subset of $\langle R \cup I \rangle$. Then Ann(P) is a pseudo neutrosophic ideal of $\langle R \cup I \rangle$.*

Definition 2.33 *Let $\langle R \cup I \rangle$ be a neutrosophic ring and let P be a neutrosophic ideal of $\langle R \cup I \rangle$. The set*

$$\langle R \cup I \rangle /P = \{r + P : r \in \langle R \cup I \rangle\}$$

is called the neutrosophic quotient ring provided that $\langle R \cup I \rangle /P$ is a neutrosophic ring.

To show that $\langle R \cup I \rangle /P$ is a neutrosophic ring, let $x = r_1 + P$ and $y = r_2 + P$ be any two elements of $\langle R \cup I \rangle /P$ and let + and . be two binary operations defined on $\langle R \cup I \rangle /P$ by:

$$\begin{aligned} x + y &= (r_1 + r_2) + P, \\ xy &= (r_1 r_2) + P, \quad r_1, r_2 \in \langle R \cup I \rangle . \end{aligned}$$

It can easily be shown that

(i) the two operations are well defined;

(ii) $(\langle R \cup I \rangle /P, +)$ is an abelian group;

(iii) $(\langle R \cup I \rangle /P, .)$ is a semigroup, and

(iv) if $z = r_3 + P$ is another element of $\langle R \cup I \rangle /P$ with $r_3 \in \langle R \cup I \rangle$, then we have $z(x + y) = zx + zy$ and $(x + y)z = xz + yz$. Accordingly, $\langle R \cup I \rangle /P$ is a neutrosophic ring with P as an additive identity element.

Definition 2.34 *Let $\langle R \cup I \rangle$ be a neutrosophic ring and let P be a neutrosophic ideal of $\langle R \cup I \rangle$. $\langle R \cup I \rangle /P$ is called a false neutrosophic quotient ring if $\langle R \cup I \rangle /P$ is just a ring and not a neutrosophic ring.*

Definition 2.35 *Let $\langle R \cup I \rangle$ be a neutrosophic ring and let P be a pseudo neutrosophic ideal of $\langle R \cup I \rangle$. $\langle R \cup I \rangle /P$ is called a pseudo neutrosophic quotient ring if $\langle R \cup I \rangle /P$ is a neutrosophic ring. If $\langle R \cup I \rangle /P$ is just a ring, then we call $\langle R \cup I \rangle /P$ a false pseudo neutrosophic quotient ring.*

Definition 2.36 *Let $\langle R \cup I \rangle$ and $\langle S \cup I \rangle$ be any two neutrosophic rings. The mapping ϕ : $\langle R \cup I \rangle \to \langle S \cup I \rangle$ is called a neutrosophic ring homomorphism if the following conditions hold:*

(i) *ϕ is a ring homomorphism;*

(ii) *$\phi(I) = I$.*

The set $\{x \in \langle R \cup I \rangle : \phi(x) = 0\}$ is called the kernel of ϕ and is denoted by Kerϕ.

Theorem 2.37 *Let $\phi : \langle R \cup I \rangle \to \langle S \cup I \rangle$ be a neutrosophic ring homomorphism and let $K = Ker\phi$ be the kernel of ϕ. Then:*

(i) *K is always a subring of $\langle R \cup I \rangle$;*

(ii) *K cannot be a nuetrosophic subring of $\langle R \cup I \rangle$;*

(iii) *K cannot be an ideal of $\langle R \cup I \rangle$.*

Proof (i) It is Clear. (ii) Since $\phi(I) = I$, it follows that $I \notin K$ and the result follows. (iii) Follows directly from (ii). □

Example 2.38 Let $\langle R \cup I \rangle$ be a nuetrosophic ring and let $\phi : \langle R \cup I \rangle \to \langle R \cup I \rangle$ be a mapping defined by $\phi(r) = r \ \ \forall\, r \in \langle R \cup I \rangle$. Then ϕ is a neutrosophic ring homomorphism.

Example 2.39 Let P be a neutrosophic ideal of the neutrosophic ring $\langle R \cup I \rangle$ and let $\phi : \langle R \cup I \rangle \to \langle R \cup I \rangle / P$ be a mapping defined by $\phi(r) = r + P, \ \forall\, r \in \langle R \cup I \rangle$. Then $\forall\, r, s \in \langle R \cup I \rangle$, we have

$$\phi(r+s) = \phi(r) + \phi(s), \qquad \phi(rs) = \phi(r)\phi(s),$$

which shows that ϕ is a ring homomorphism. But then,

$$\phi(I) = I + P \neq I.$$

Thus, ϕ is not a neutrosophic ring homomorphism. This is another marked difference between the classical ring concept and the concept of netrosophic ring.

Proposition 2.40 *Let $(\langle R \cup I \rangle, +)$ be a neutrosophic abelian group and let $Hom(\langle R \cup I \rangle, \langle R \cup I \rangle)$ be the set of neutrosophic endomorphisms of $(\langle R \cup I \rangle, +)$ into itself. Let $+$ and $.$ be addition and multiplication in $Hom(\langle R \cup I \rangle, \langle R \cup I \rangle)$ defined by*

$$\begin{aligned} (\phi + \psi)(x) &= \phi(x) + \psi(x), \\ (\phi.\psi)(x) &= \phi(\psi(x)), \forall\, \phi, \psi \in Hom(\langle R \cup I \rangle, \langle R \cup I \rangle), x \in \langle R \cup I \rangle. \end{aligned}$$

Then $(Hom(\langle R \cup I \rangle, \langle R \cup I \rangle), +, .)$ is a neutrosophic ring.

Proof The proof is the same as in the classical ring. □

Definition 2.41 *Let R be an arbitrary ring with unity. A neutrosophic left R-module is a neutrosophic abelian group $(\langle M \cup I \rangle, +)$ together with a scalar multiplication map $. : R \times \langle M \cup I \rangle \to \langle M \cup I \rangle$ that satisfies the following conditions:*

(i) $r(m+n) = rm + rn$;

(ii) $(r+s)m = rm + sm$;

(iii) $(rs)m = r(sm)$;

(iv) $1.m = m$, *where* $r, s \in R$ *and* $m, n \in \langle M \cup I \rangle$.

Definition 2.42 *Let R be an arbitrary ring with unity. A neutrosophic right R-module is a neutrosophic abelian group $(\langle M \cup I \rangle, +)$ together with a scalar multiplication map $. : \langle M \cup I \rangle \times R \to \langle M \cup I \rangle$ that satisfies the following conditions:*

(i) $(m+n)r = mr + nr$;

(ii) $m(r+s) = mr + ms$;

(iii) $m(rs) = (mr)s$;

(iv) $m.1 = m$, *where* $r, s \in R$ *and* $m, n \in \langle M \cup I \rangle$.

If R is a commutative ring, then a neutrosophic left R-module $\langle M \cup I \rangle$ becomes a neutrosophic right R-module and we simply call $\langle M \cup I \rangle$ a neutrosophic R-module.

Example 2.43 Let $(\langle M \cup I \rangle, +)$ be a nuetrosophic abelian group and let $\mathcal{Z}$ be the ring of integers. If we define the mapping $f : \mathcal{Z} \times \langle M \cup I \rangle \to \langle M \cup I \rangle$ by $f(n, m) = nm, \forall\, n \in \mathcal{Z}, m \in \langle M \cup I \rangle$, then $\langle M \cup I \rangle$ becomes a neutrosophic $\mathcal{Z}$-module.

Example 2.44 Let $\langle R \cup I \rangle [x]$ be a neutrosophic ring of polynomials where R is a commutative ring with unity. Obviously, $(\langle R \cup I \rangle [x], +)$ is a neutrosophic abelian group and the scalar multiplication map $. : R \times \langle R \cup I \rangle [x] \to \langle R \cup I \rangle [x]$ satisfies all the axioms of the neutrosophic R-module. Hence, $\langle R \cup I \rangle [x]$ is a neutrosophic R-module.

Proposition 2.45 *Let $(\langle R \cup I \rangle, +)$ be a neutrosophic abelian group and let $Hom(\langle R \cup I \rangle, \langle R \cup I \rangle)$ be the neutrosophic ring obtained in Proposition (2.40). Let $. : Hom(\langle R \cup I \rangle, \langle R \cup I \rangle) \times \langle R \cup I \rangle \to \langle R \cup I \rangle$ be a scalar multiplication defined by $.(f, r) = fr, \ \forall\, f \in Hom(\langle R \cup I \rangle, \langle R \cup I \rangle)$, $r \in \langle R \cup I \rangle$. Then $\langle R \cup I \rangle$ is a neutrosophic left $Hom(\langle R \cup I \rangle, \langle R \cup I \rangle)$-module.*

Proof Suppose that $\mathrm{Hom}(\langle R \cup I \rangle, \langle R \cup I \rangle)$ is a neutrosophic ring. Then by Theorem (2.3), it is also a ring. It is clear that $.(f, r) = fr$ is the image of r under f and it is an element of $\langle R \cup I \rangle$. It can easily be shown that the scalar multiplication "." satisfies the axioms of a neutrosophic left R-module. Hence, $\langle R \cup I \rangle$ is a neutrosophic left $\mathrm{Hom}(\langle R \cup I \rangle, \langle R \cup I \rangle)$-module. □

Definition 2.46 *Let $\langle M \cup I \rangle$ be a neutrosophic left R-module. The set $\{r \in R : rm = 0 \ \ \forall\, m \in \langle M \cup I \rangle\}$ is called the annihilator of $\langle M \cup I \rangle$ and is denoted by $Ann(\langle M \cup I \rangle)$. $\langle M \cup I \rangle$ is said to be faithful if $Ann(\langle M \cup I \rangle) = (0)$. It can easily be shown that $Ann(\langle M \cup I \rangle)$ is a pseudo neutrosophic ideal of $\langle M \cup I \rangle$.*

§3. Neutrosophic Polynomial Rings

In this section and Sections 4 and 5, unless otherwise stated, all neutrosophic rings will be assumed to be commutative neutrosophic rings with unity and x will be an indetrminate in $\langle R \cup I \rangle [x]$.

Definition 3.1 (i) *By the neutrosophic polynomial ring in x denoted by $\langle R \cup I \rangle [x]$ we mean the set of all symbols $\sum_{i=1}^{n} a_i x^i$ where n can be any nonnegative integer and where the coefficients $a_i, \ i = n, n-1, \ldots, 2, 1, 0$ are all in $\langle R \cup I \rangle$.*

(ii) *If $f(x) = \sum_{i=1}^{n} a_i x^i$ is a neutrosophic polynomial in $\langle R \cup I \rangle [x]$ such that $a_i = 0, \ \forall\, i = n, n-1, \ldots, 2, 1, 0$, then we call f(x) a zero neutrosophic polynomial in $\langle R \cup I \rangle [x]$.*

(iii) *If $f(x) = \sum_{i=1}^{n} a_i x^i$ is a nonzero neutrosophic polynomial in $\langle R \cup I \rangle [x]$ with $a_n \neq 0$, then we call n the degree of f(x) denoted by deg f(x) and we write $deg f(x) = n$.*

(iv) *Two neutrosophic polynomials $f(x) = \sum_{i=1}^{n} a_i x^i$ and $g(x) = \sum_{j=1}^{m} b_j x^j$ in $\langle R \cup I \rangle [x]$ are said to be equal written $f(x) = g(x)$ if and only if for every integer $i \geq 0$, $a_i = b_i$ and $n = m$.*

(v) *A neutrosophic polynomial $f(x) = \sum_{i=1}^{n} a_i x^i$ in $\langle R \cup I \rangle [x]$ is called a strong neutrosophic polynomial if for every $i \geq 0$, each a_i is of the form $(a + bI)$ where $a, b \in R$ and $b \neq 0$.*

$f(x)=\sum_{i=1}^{n} a_i x^i$ is called a mixed neutrosophic polynomial if some $a_i \in R$ and some a_i are of the form $(a+bI)$ with $b \neq 0$. If every $a_i \in R$ then $f(x)=\sum_{i=1}^{n} a_i x^i$ is called a polynomial.

Example 3.2 $\langle \mathcal{Z} \cup I\rangle [x], \langle \mathcal{Q} \cup I\rangle [x], \langle \mathcal{R} \cup I\rangle [x], \langle \mathcal{C} \cup I\rangle [x]$ are neutrosophic polynomial rings of integers, rationals, real and complex numbers respectively each of zero characteristic.

Example 3.3 Let $\langle \mathcal{Z}_n \cup I\rangle$ be the neutrosophic ring of integers modulo n. Then $\langle \mathcal{Z}_n \cup I\rangle [x]$ is the neutrosophic polynomial ring of integers modulo n. The characteristic of $\langle \mathcal{Z}_n \cup I\rangle [x]$ is n. If $n=3$ and $\langle \mathcal{Z}_3 \cup I\rangle [x]=\{ax^2+bx+c : a,b,c \in \langle \mathcal{Z}_3 \cup I\rangle\}$, then $\langle \mathcal{Z}_3 \cup I\rangle [x]$ is a neutrosophic polynomial ring of integers modulo 3.

Example 3.4 Let $f(x), g(x) \in \langle \mathcal{Z} \cup I\rangle [x]$ such that $f(x)=2Ix^2+(2+I)x+(1-2I)$ and $g(x)=x^3-(1-3I)x^2+3Ix+(1+I)$. Then f(x) and g(x) are strong and mixed neutrosophic polynomials of degrees 2 and 3 respectively.

Definition 3.5 *Let α be a fixed element of the neutrosophic ring $\langle R \cup I\rangle$. The mapping $\phi_\alpha : \langle R \cup I\rangle [x] \to \langle R \cup I\rangle$ defined by*

$$\phi_\alpha \left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right) = a_n \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0$$

is called the neutrosophic evaluation map. It can be shown that ϕ_α is a neutrosophic ring homomorphism. If $R=\mathcal{Z}$ and $f(x) \in \langle \mathcal{Z} \cup I\rangle [x]$ such that $f(x)=2Ix^2+x-3I$, then $\phi_{1+I}(f(x))=1+6I$ and $\phi_I(f(x))=0$. The last result shows that $f(x)$ is in the kernel of ϕ_I.

Theorem 3.6([1]) *Every neutrosophic polynomial ring $\langle R \cup I\rangle [x]$ contains a polynomial ring R[x].*

Theorem 3.7 *The neutrosophic ring $\langle R \cup I\rangle$ is not an integral domain (ID) even if R is an ID.*

Proof Suppose that $\langle R \cup I\rangle$ is an ID. Obviously, $R \subset \langle R \cup I\rangle$. Let $x=(\alpha-\alpha I)$ and $y=\beta I$ be two elements of $\langle R \cup I\rangle$ where α and β are non-zero positive integers. Clearly, $x \neq 0$ and $y \neq 0$ and since $I^2=I$, we have $xy=0$ which shows that x and y are neutrosophic zero divisors in $\langle R \cup I\rangle$ and consequently, $\langle R \cup I\rangle$ is not an ID. □

Theorem 3.8 *The neutrosophic polynomial ring $\langle R \cup I\rangle [x]$ is not an ID even if R is an ID.*

Proof Suppose that R is an ID. Then $R[x]$ is also an ID and $R[x] \subset \langle R \cup I\rangle [x]$. But then by Theorem 3.7, $\langle R \cup I\rangle$ is not an ID and therefore $\langle R \cup I\rangle [x]$ cannot be an ID. □

Example 3.9 Let $\langle \mathcal{Z} \cup I\rangle [x]$ be the neutrosophic polynomial ring of integers and let f(x), g(x), p(x)and q(x) be neutrosophic polynomials in $\in \langle \mathcal{Z} \cup I\rangle$ given by $f(x)=(2-2I)x^2+3Ix-I$, $g(x)=Ix+(1+I)$, $p(x)=(8-8I)x^5$ and $q(x)=7Ix^3$. Then $f(x)g(x)=(2+I)x^2+5Ix-2I$ and $p(x)q(x)=0$. Now $deg\ f(x)+deg\ g(x)=3$, $deg(f(x)g(x))=2<3$, $deg\ p(x)+deg\ q(x)=8$ and $deg(p(x)q(x))=0<8$. The causes of these phenomena are the existence of neutrosophic zero divisors in $\langle \mathcal{Z} \cup I\rangle$ and $\langle \mathcal{Z} \cup I\rangle [x]$ respectively. We register these observations in the following theorem.

Theorem 3.10 *Let $\langle R \cup I \rangle$ be a commutative neutrosophic ring with unity. If $f(x) = \sum_{i=1}^{n} a_i x^i$ and $g(x) = \sum_{j=1}^{m} b_j x^j$ are two non-zero neutrosophic polynomials in $\langle R \cup I \rangle [x]$ with R an ID or not such that $a_n = (\alpha - \alpha I)$ and $b_m = \beta I$ where α and β are non-zero positive integers, then*

$$deg(f(x)g(x)) < deg\ f(x) + deg\ g(x).$$

Proof Suppose that $f(x) = \sum_{i=1}^{n} a_i x^i$ and $g(x) = \sum_{j=1}^{m} b_j x^j$ are two non-zero neutrosophic polynomials in $\langle R \cup I \rangle [x]$ with $a_n = (\alpha - \alpha I)$ and $b_m = \beta I$ where α and β are non-zero positive integers. Clearly, $a_n \neq 0$ and $b_m \neq 0$ but then $a_n b_m = 0$ and consequently,

$$\begin{aligned} deg(f(x)g(x)) &= (n-1) + (m-1) \\ &= (n+m) - 2 < (n+m) \\ &= deg\ f(x) + deg\ g(x). \end{aligned}$$

□

§4. Factorization in Neutrosophic Polynomial Rings

Definition 4.1 *Let $f(x) \in \langle R \cup I \rangle [x]$ be a neutrosophic polynomial. Then*

(i) f(x) is said to be neutrosophic reducible in $\langle R \cup I \rangle [x]$ if there exits two neutrosophic polynomials $p(x), q(x) \in \langle R \cup I \rangle [x]$ such that $f(x) = p(x).q(x)$.

(ii) f(x) is said to be semi neutrosophic reducible if $f(x) = p(x).q(x)$ but only one of p(x) or q(x) is a neutrosophic polynomial in $\langle R \cup I \rangle [x]$.

(iii) f(x) is said to be neutrosophic irreducible if $f(x) = p(x).q(x)$ but either p(x) or q(x) equals I or 1.

Definition 4.2 *Let f(x) and g(x) be two neutrosophic polynomials in the neutrosophic polynomial ring $\langle R \cup I \rangle [x]$. Then*

(i) The pair f(x) and g(x) are said to be relatively neutrosophic prime if the gcd $(f(x), g(x)) = r(x)$ is not possible for a neutrosophic polynomial $r(x) \in \langle R \cup I \rangle [x]$.

(ii) The pair f(x) and g(x) are said to be strongly relatively neutrosophic prime if their gcd $(f(x), g(x)) = 1$ or I.

Definition 4.3 *A neutrosophic polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \langle \mathcal{Z} \cup I \rangle [x]$ is said to be neutrosophic primitive if the gcd $(a_n, a_{n-1}, \cdots, a_1, a_0) = 1$ or I.*

Definition 4.3 *Let $f(x) = \sum_{i=1}^{n} a_i x^i$ be a neutrosophic polynomial in $\langle R \cup I \rangle [x]$. f(x) is said to be a neutrosophic monic polynomial if $a_n = 1$.*

Example 4.5 Let us consider the neutrosophic polynomial ring $\langle \mathcal{R} \cup I \rangle [x]$ of all real numbers and let f(x) and d(x) be two neutrosophic polynomials in $\langle \mathcal{R} \cup I \rangle [x]$.

(i) If $f(x) = 2Ix^2 - (1 + 7I)x + 6I$ and $d(x) = x - 3I$, then by dividing f(x) by d(x) we obtain the quotient $q(x) = 2Ix - (1 + I)$ and the remainder $r(x) = 0$ and hence $f(x) \equiv (2Ix - (1 + I))(x - 3I) + 0$.

(*ii*) If $f(x) = 2Ix^3 + (1 + I)$ and $d(x) = Ix + (2 - I)$, then $q(x) = 2Ix^2 - 2Ix + 2I)$, $r(x) = 1 - I$ and $f(x) \equiv (2Ix^2 - 2Ix + 2I))(Ix + (2 - I)) + (1 - I)$.

(*iii*) If $f(x) = (2+I)x^2+2Ix+(1+I)$ and $d(x) = (2+I)x+(2-I)$, then $q(x) = x-\left(1 - \frac{4}{3}I\right)$, $r(x) = 3 - \frac{4}{3}I$ and $f(x) \equiv \left(x - \left(1 - \frac{4}{3}\right)\right)((2 + I)x - (2 - I)) + \left(3 - \frac{4}{3}I\right)$.

(*iv*) If $f(x) = Ix^2 + x - (1+5I)$ and $d(x) = x - (1+I)$, then $q(x) = Ix+(1+2I)$, $r(x) = 0$ and $f(x) \equiv (Ix + (1 + 2I))(x - (1 + I)) + 0$.

(*v*) If $f(x) = x^2 - Ix + (1 + I)$ and $d(x) = x - (1 - I)$, then $q(x) = x + (1 - 2I)$, $r(x) = 2$ and $f(x) \equiv (x + (1 - 2I))(x - (1 - I)) + 2$.

The examples above show that for each pair of the neutrosophic polynomials f(x) and d(x) considered there exist unique neutrosophic polynomials $q(x), r(x) \in \langle \mathcal{R} \cup I \rangle [x]$ such that $f(x) = q(x)d(x) + r(x)$ where *deg* $r(x) <$ *deg* $d(x)$. However, this is generally not true. To see this let us consider the following pairs of neutrosophic polynomials in $\langle \mathcal{R} \cup I \rangle [x]$:

(*i*) $f(x) = 4Ix^2 + (1 + I)x - 2I, d(x) = 2Ix + (1 + I)$;

(*ii*) $f(x) = 2Ix^2 + (1 + I)x + (1 - I), d(x) = 2Ix + (3 - 2I)$;

(*iii*) $f(x) = (-8I)x^2 + (7 + 5I)x + (2 - I), d(x) = Ix + (1 + I)$;

(*iv*) $f(x) = Ix^2 - 2Ix + (1 + I), d(x) = Ix - (1 - I)$.

In each of these examples, it is not possible to find $q(x), r(x) \in \langle \mathcal{R} \cup I \rangle [x]$ such that $f(x) = q(x)d(x) + r(x)$ with *deg* $r(x) <$ *deg* $d(x)$. Hence Division Algorithm is generally not possible for neutrosophic polynomial rings. However for neutrosophic polynomial rings in which all neutrosophic polynomials are neutrosophic monic, the Division Algorithm holds generally. The question of wether Division Algorithm is true for neutrosophic polynomial rings raised by Vasantha Kandasamy and Florentin Smarandache in [1] is thus answered.

Theorem 4.6 *If $f(x)$ and $d(x)$ are neutrosophic polynomials in the neutrosophic polynomial ring $\langle R \cup I \rangle [x]$ with f(x) and d(x) neutrosophic monic, there exist unique neutrosophic polynomials $q(x), r(x) \in \langle R \cup I \rangle [x]$ such that $f(x) = q(x)d(x) + r(x)$ with deg $r(x) <$ deg $d(x)$.*

Proof The proof is the same as the classical case. □

Theorem 4.7 *Let $f(x)$ be a neutrosophic monic polynomial in $\langle R \cup I \rangle [x]$ and for $u \in \langle R \cup I \rangle$, let $d(x) = x - u$. Then $f(u)$ is the remainder when $f(x)$ is divided by $d(x)$. Furthermore, if $f(u) = 0$ then $d(x)$ is a neutrosophic factor of $f(x)$.*

Proof Since $f(x)$ and $d(x)$ are neutrosophic monic in $\langle R \cup I \rangle [x]$, there exists $q(x)$ and $r(x)$ in $\langle R \cup I \rangle [x]$ such that $f(x) = q(x)(x - u) + r(x)$, with $r(x) = 0$ or *deg* $r(x) <$ *deg* $d(x) = 1$. Hence $r(x) = r \in \langle R \cup I \rangle$. Now, $\phi_u(f(x)) = 0 + r(u) = r(u) = r \in \langle R \cup I \rangle$. If $f(u) = 0$, it follows that $r(x) = 0$ and consequently, $d(x)$ is a neutrosophic factor of $f(x)$. □

Observation 4.8 Since the indeterminancy factor I has no inverse, it follows that the neutrosophic rings $\langle \mathcal{Q} \cup I \rangle, \langle \mathcal{R} \cup I \rangle, \langle \mathcal{C} \cup I \rangle$ cannot be neutrosophic fields and consequently neutrosophic equations of the form $(a + bI)x = (c + dI)$ are not solvable in $\langle \mathcal{Q} \cup I \rangle, \langle \mathcal{R} \cup I \rangle, \langle \mathcal{C} \cup I \rangle$ except $b \equiv 0$.

Definition 4.9 *Let $f(x)$ be a neutrosophic polynomial in $\langle R \cup I \rangle [x]$ with deg $f(x) \geq 1$. An*

element $u \in \langle R \cup I\rangle$ is said to be a neutrosophic zero of f(x) if $f(u) = 0$.

Example 4.10 (*i*) Let $f(x) = 6x^2 + Ix - 2I \in \langle \mathcal{Q} \cup I\rangle [x]$. Then f(x) is neutrosophic reducible and (2x-I) and (3x+2I) are the neutrosophic factors of $f(x)$. Since $f(\frac{1}{2}I) = 0$ and $f(-\frac{2}{3}I) = 0$, then $\frac{1}{2}I, -\frac{2}{3}I \in \langle \mathcal{Q} \cup I\rangle$ are the neutrosophic zeroes of $f(x)$. Since $f(x)$ is of degree 2 and it has two zeroes, then the Fundamental Theorem of Algebra is obeyed.

(*ii*) Let $f(x) = 4Ix^2 + (1 + I)x - 2I \in \langle \mathcal{Q} \cup I\rangle [x]$. $f(x)$ is neutrosophic reducible and $p(x) = 2Ix + (1 + I)$ and $q(x) = (1 + I)x - I$ are the neutrosophic factors of $f(x)$. But then, $f(x)$ has no neutrosophic zeroes in $\langle \mathcal{Q} \cup I\rangle$ and even in $\langle \mathcal{R} \cup I\rangle$ and $\langle \mathcal{C} \cup I\rangle$ since I^{-1}, the inverse of I does not exist.

(*iii*) $Ix^2 - 2$ is neutrosophic irreducible in $\langle \mathcal{Q} \cup I\rangle [x]$ but it is neutrosophic reducible in $\langle \mathcal{R} \cup I\rangle [x]$ since $Ix^2 - 2 = (Ix - \sqrt{2})(Ix + \sqrt{2})$. However since $\langle \mathcal{R} \cup I\rangle$ is not a field, $Ix^2 - 2$ has no neutrosophic zeroes in $\langle \mathcal{R} \cup I\rangle$.

Theorem 4.11 *Let $f(x)$ be a neutrosophic polynomial of degree > 1 in the neutrosophic polynomial ring $\langle R \cup I\rangle [x]$. If $f(x)$ has neutrosophic zeroes in $\langle R \cup I\rangle$, then $f(x)$ is neutrosophic reducible in $\langle R \cup I\rangle [x]$ and not the converse.*

Theorem 4.12 *Let $f(x)$ be a neutrosophic polynomial in $\langle R \cup I\rangle [x]$. The factorization of $f(x)$ if possible over $\langle R \cup I\rangle [x]$ is not unique.*

Proof Let us consider the neutrosophic polynomial $f(x) = 2Ix^2 + (1 + I)x + 2I$ in the neutrosophic ring of polynomials $\langle \mathcal{Z}_3 \cup I\rangle [x]$. $f(I) = 0$ and by Theorem 4.11, $f(x)$ is neutrosophic reducible in $\langle \mathcal{Z}_3 \cup I\rangle [x]$ and hence f(x) can be expressed as $f(x) = (2Ix + 1)(x - I) = (2Ix + 1)(x + 2I)$. However, $f(x)$ can also be expressed as $f(x) = [(1 + I)x + I][Ix + (1 + I)]$. This shows that the factorization of $f(x)$ is not unique in $\langle \mathcal{Z}_3 \cup I\rangle [x]$. We note that the first factorization shows that $f(x)$ has $I \in \langle \mathcal{Z}_3 \cup I\rangle$ as a neutrosophic zero but the second factorization shows that f(x) has no neutrosophic zeroes in $\langle \mathcal{Z}_3 \cup I\rangle$. This is different from what obtains in the classical rings of polynomials. □

Observation 4.13 Let us consider the neutrosophic polynomial ring $\langle R \cup I\rangle [x]$. It has been shown in Theorem 3.8 that $\langle R \cup I\rangle [x]$ cannot be a neutrosophic ID even if R is an ID. Also by Theorem 4.12, factorization in $\langle R \cup I\rangle [x]$ is generally not unique. Consequently, $\langle R \cup I\rangle [x]$ cannot be a neutrosophic Unique Factorization Domain (UFD) even if R is a UFD. Thus Gauss's Lemma, which asserts that R[x] is a UFD if and only if R is a UFD does not hold in the setting of neutrosophic polynomial rings. Also since $I \in \langle R \cup I\rangle$ and I^{-1}, the inverse of I does not exist, then $\langle R \cup I\rangle$ cannot be a field even if R is a field and consequently $\langle R \cup I\rangle [x]$ cannot be a neutrosophic UFD. Again, the question of wether $\langle R \cup I\rangle [x]$ is a neutrosophic UFD given that R is a UFD raised by Vasantha Kandasamy and Florentin Smarandache in [1] is answered.

§5. Neutrosophic Ideals in Neutrosophic Polynomial Rings

Definition 5.1 *Let $\langle R \cup I\rangle [x]$ be a neutrosophic ring of polynomials. An ideal J of $\langle R \cup I\rangle [x]$*

is called a neutrosophic principal ideal if it can be generated by an irreducible neutrosophic polynomial $f(x)$ in $\langle R \cup I \rangle [x]$.

Definition 5.2 *A neutrosophic ideal P of a neutrosophic ring of polynomials $\langle R \cup I \rangle [x]$ is called a neutrosophic prime ideal if $f(x)g(x) \in P$, then $f(x) \in P$ or $g(x) \in P$ where $f(x)$ and $g(x)$ are neutrosophic polynomials in $\langle R \cup I \rangle [x]$.*

Definition 5.3 *A neutrosophic ideal M of a neutrosophic ring of polynomials $\langle R \cup I \rangle [x]$ is called a neutrosophic maximal ideal of $\langle R \cup I \rangle [x]$ if $M \neq \langle R \cup I \rangle [x]$ and no proper neutrosophic ideal N of $\langle R \cup I \rangle [x]$ properly contains M that is if $M \subseteq N \subseteq \langle R \cup I \rangle [x]$ then $M = N$ or $N = \langle R \cup I \rangle [x]$.*

Example 5.4 Let $\langle \mathcal{Z}_2 \cup I \rangle [x] = \{ax^2 + bx + c : a, b, c \in \langle \mathcal{Z}_2 \cup I \rangle\}$ and consider $f(x) = Ix^2 + Ix + (1 + I) \in \langle \mathcal{Z}_2 \cup I \rangle [x]$. The neutrosophic ideal $J =< f(x) >$ generated by $f(x)$ is neither a neutrosophic principal ideal nor a neutrosophic prime ideal of $\langle \mathcal{Z}_2 \cup I \rangle [x]$. This is so because f(x) is neutrosophic reducible in $\langle \mathcal{Z}_2 \cup I \rangle [x]$ eventhough it does not have zeroes in $\langle \mathcal{Z}_2 \cup I \rangle$. Also, $(Ix + (1 + I))(Ix + 1) \in J$ but $(Ix + (1 + I)) \notin J$ and $(Ix + 1) \notin J$. Hence J is not a neutrosophic prime ideal of $\langle \mathcal{Z}_2 \cup I \rangle [x]$. However, $< 0 >$ is the only neutrosophic prime ideal of $\langle \mathcal{Z}_2 \cup I \rangle [x]$ which is not a neutrosophic maximal ideal.

Theorem 5.5 *Let $\langle R \cup I \rangle [x]$ be a neutrosophic ring of polynomials. Every neutrosophic principal ideal of $\langle R \cup I \rangle [x]$ is not prime.*

Proof Consider the neutrosophic polynomial ring $\langle \mathcal{Z}_3 \cup I \rangle [x] = \{x^3 + ax + b : a, b \in \langle \mathcal{Z}_3 \cup I \rangle\}$ and Let $f(x) = x^3 + Ix + (1 + I)$. It can be shown that $f(x)$ is neutrosophic irreducible in $\langle \mathcal{Z}_3 \cup I \rangle [x]$ and therefore $< f(x) >$, the neutrosophic ideal generated by f(x) is principal and not a prime ideal. We have also answered the question of Vasantha Kandasamy and Florentin Smarandache in [1] of wether every neutrosophic principal ideal of $\langle R \cup I \rangle [x]$ is also a neutrosophic prime ideal. □

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Divisor Cordial Graphs

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Abstract: A divisor cordial labeling of a graph G with vertex set V is a bijection f from V to $\{1, 2, \cdots, |V|\}$ such that an edge uv is assigned the label 1 if $f(u)$ divides $f(v)$ or $f(v)$ divides $f(u)$ and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. If a graph has a divisor cordial labeling, then it is called divisor cordial graph. In this paper, we proved the standard graphs such as path, cycle, wheel, star and some complete bipartite graphs are divisor cordial. We also proved that complete graph is not divisor cordial.

Key Words: Cordial labeling, divisor cordial labeling, divisor cordial graph

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§1. Introduction

By a graph, we mean a finite, undirected graph without loops and multiple edges, for terms not defined here, we refer to Harary [5].

First we give the some concepts in number theory [3].

Definition 1.1 *Let a and b be two integers. If a divides b means that there is a positive integer k such that $b = ka$. It is denoted by $a \mid b$.*

If a does not divide b, then we denote $a \nmid b$.

Now we give the definition of divisor function.

Definition 1.2 *The divisor function of integer $d(n)$ is defined by $d(n)=\Sigma 1$. That is, $d(n)$ denotes the number of divisor of an integer n.*

Next we define the divisor summability function.

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Definition 1.3 *Let n be an integer and x be a real number. The divisor summability function is defined as $D(x) = \Sigma d(n)$. That is, $D(x)$ is the sum of the number of divisor of n for $n \leqslant x$.*

The big O notation is defined as follows.

Definition 1.4 *Let $f(x)$ and $g(x)$ be two functions defined on some subset of the real numbers. $f(x) = O(g(x))$ as $x \to \infty$ if and only if there is a positive real number M and a real number x_0 such that $|\ f(x)| \leqslant M\ |\ g(x)|$ for all $x > x_0$.*

Next, we state Dirichlet's divisor result as follows.

Result 1.5 *$D(x) = x\log x + x(2\gamma - 1) + \Delta(x)$ where γ is the Euler-Mascheroni Constant given by γ=0.577 approximately and $\Delta(x) = O(\sqrt{x})$.*

Graph labeling [4] is a strong communication between number theory [3] and structure of graphs [5]. By combining the divisibility concept in number theory and cordial labeling concept in Graph labeling, we introduce a new concept called divisor cordial labeling. In this paper, we prove the standard graphs such as path, cycle, wheel, star and some complete bipartite graphs are divisor cordial and complete graph is not divisor cordial.

A vertex labeling [4] of a graph G is an assignment f of labels to the vertices of G that induces for each edge uv a label depending on the vertex label $f(u)$ and $f(v)$. The two best known labeling methods are called graceful and harmonious labelings. Cordial labeling is a variation of both graceful and harmonious labelings [1].

Definition 1.6 *Let $G = (V, E)$ be a graph. A mapping $f : V(G) \to \{0, 1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f.*

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0, 1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and $e_f(0)$, $e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

The concept of cordial labeling was introduced by Cahit [1] and he got some results in [2].

Definition 1.7 *A binary vertex labeling of a graph G is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is cordial if it admits cordial labeling*

§2. Main Results

Sundaram, Ponraj and Somasundaram [6] have introduced the notion of prime cordial labeling.

Definition 2.1([6]) *A prime cordial labeling of a graph G with vertex set V is a bijection f from V to $\{1, 2, \ldots, |V|\}$ such that if each edge uv assigned the label 1 if $\gcd(f(u), f(v)) = 1$ and 0 if $\gcd(f(u), f(v)) > 1$, then the number of edges labeled with 1 and the number of edges labeled with 0 differ by at most 1.*

In [6], they have proved some graphs are prime cordial. Motivated by the concept of

prime cordial labeling, we introduce a new special type of cordial labeling called divisor cordial labeling as follows.

Definition 2.2 *Let $G = (V, E)$ be a simple graph and $f : V \to \{1, 2, \ldots |V|\}$ be a bijection. For each edge uv, assign the label 1 if either $f(u) \mid f(v)$ or $f(v) \mid f(u)$ and the label 0 if $f(u) \nmid f(v)$. f is called a divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1$.*

A graph with a divisor cordial labeling is called a divisor cordial graph.

Example 2.3 Consider the following graph G.

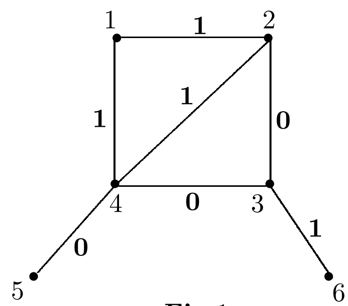

Fig.1

We see that $e_f(0)$= 3 and $e_f(1)$= 4. Thus $|e_f(0) - e_f(1)| = 1$ and hence G is divisor cordial.

Theorem 2.4 *The path P_n is divisor cordial.*

Proof Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n. Label these vertices in the following order.

$$\begin{array}{ccccc}
1, & 2, & 2^2, & \ldots, & 2^{k_1}, \\
3, & 3 \times 2, & 3 \times 2^2, & \ldots, & 3 \times 2^{k_2}, \\
5, & 5 \times 2, & 5 \times 2^2, & \ldots, & 5 \times 2^{k_3}, \\
\cdots & \cdots & \cdots & \cdots & \ldots, \\
\cdots & \cdots & \cdots & \cdots & \ldots .
\end{array}$$

where $(2m-1)2^{k_m} \leq n$ and $m \geq 1$, $k_m \geq 0$. We observe that $(2m-1)2^a$ divides $(2m-1)2^b$ $(a < b)$ and $(2m-1)2^{k_i}$ does not divide $2m+1$.

In the above labeling, we see that the consecutive adjacent vertices having the labels even numbers and consecutive adjacent vertices having labels odd and even numbers contribute 1 to each edge. Similarly, the consecutive adjacent vertices having the labels odd numbers and consecutive adjacent vertices having labels even and odd numbers contribute 0 to each edge.

Thus, $e_f(1) = \dfrac{n}{2}$ and $e_f(0) = \dfrac{n-2}{2}$ if n is even and $e_f(1) = e_f(0) = \dfrac{n-1}{2}$ if n is odd. Hence $|e_f(0) - e_f(1)| \leq 1$. Thus, P_n is divisor cordial. □

Theorem 2.4 can be illustrated in the following example.

Example 2.5 (1) n is even. Particularly, let $n = 12$.

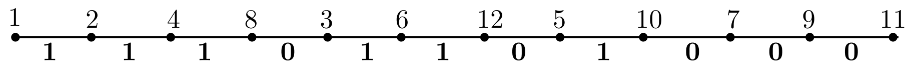

Fig.2

Here $e_f(1) = 6$ and $e_f(0) = 5$. Hence $|e_f(0) - e_f(1)| = 1$.

(2) n is odd. Particularly, let $n = 11$.

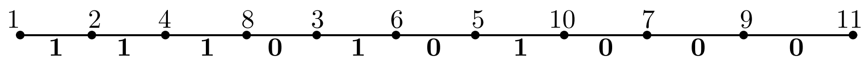

Fig.3

Here $e_f(0) = e_f(1) = 5$ and $|e_f(0) - e_f(1)| = 0$.

Observation 2.6 In the above labeling of path,

(1) the labels of vertices v_1 and v_2 must be 1 and 2 respectively, for all n. and

(2) the label of last vertex is always an odd number for $n \geq 3$.

In particular, the label v_n is n or $n-1$ according as n is odd or even.

Theorem 2.7 *The cycle C_n is divisor cordial.*

Proof Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle C_n. We follow the same labeling pattern as in the path, except by interchanging the labels of v_1 and v_2. Then it follows from the observation (2). Thus C_n is divisor cordial. □

Theorem 2.8 *The wheel graph $W_n = K_1 + C_{n-1}$ is divisor cordial.*

Proof Let v_1 be the central vertex and $v_2, v_3, \ldots, v_n$ be the vertices of C_{n-1}.

Case 1 n is odd.

Label the vertices $v_1, v_2, \ldots, v_n$ as in the labels of cycle C_n in the Theorem 2.7, with the same order.

Case 2 n is even.

Label the vertices $v_1, v_2, \ldots, v_n$ as in the labels of cycle in the Theorem 2.7, with the same order except by interchanging the labels of the vertices v_2 and v_3.

In both the cases, we see that $e_f(0) = e_f(1) = n-1$. Hence $|e_f(0) - e_f(1)| = 0$. Thus, W_n is divisor cordial. □

The labeling pattern in the Theorem 2.8 is illustrated in the following example.

Example 2.9 (1) n is odd. Particularly, let $n = 11$.

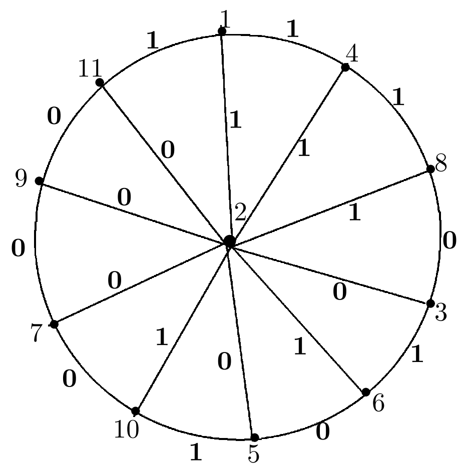

Fig.4

We see that $e_f(0) = e_f(1) = 10$.

(2) n is even. Particularly, let $n = 14$.

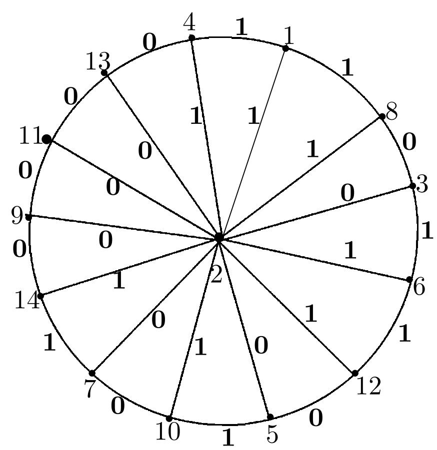

Fig.5

We see that $e_f(0) = e_f(1) = 13$.

Now we discuss the divisor cordiality of complete bipartite graphs.

Theorem 2.10 *The star graph $K_{1,n}$ is divisor cordial.*

Proof Let v be the central vertex and let $v_1, v_2, \ldots, v_n$ be the end vertices of the star $K_{1,n}$. Now assign the label 2 to the vertex v and the remaining labels to the vertices $v_1, v_2, \ldots, v_n$.

Then we see that

$$e_f(0) - e_f(1) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

Thus $|e_f(0) - e_f(1)| \leq 1$ and hence $K_{1,n}$ is divisor cordial. □

Theorem 2.11 *The complete bipartite graph $K_{2,n}$ is divisor cordial.*

Proof Let $V = V_1 \cup V_2$ be the bipartition of $K_{2,n}$ such that $V_1 = \{x_1, x_2\}$ and $V_2 = \{y_1, y_2, \ldots, y_n\}$. Now assign the label 1 to x_1 and the largest prime number p such that $p \leq n+2$ to x_2 and the remaining labels to the vertices $y_1, y_2, \ldots, y_n$. Then it follows that $e_f(0) = e_f(1) = n$ and hence $K_{2,n}$ is divisor cordial. □

Theorem 2.12 *The complete bipartite graph $K_{3,n}$ is divisor cordial.*

Proof Let $V = V_1 \cup V_2$ be the bipartition of V such that $V_1 = \{x_1, x_2, x_3\}$ and $V_2 = \{y_1, y_2, \ldots, y_n\}$. Now define $f(x_1) = 1$, $f(x_2) = 2$, $f(x_3) = p$, where p is the largest prime number such that $p \leq n+3$ and the remaining labels to the vertices $y_1, y_2, \ldots, y_n$. Then clearly

$$e_f(0) - e_f(1) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Thus, $K_{3,n}$ is divisor cordial. □

Next we are trying to investigate the divisor cordiality of K_n. Obviously, K_1,K_2 and K_3 are divisor cordial. Now we consider K_4. The labeling of K_4 is given as follows.

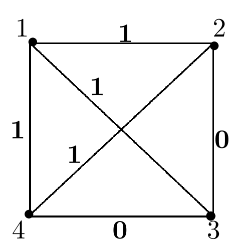

Fig.6

We see that $|e_f(0) - e_f(1)| = 2$ and hence K_4 is not divisor cordial. Next, we consider K_5.

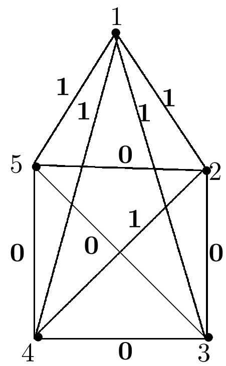

Fig.7

Here $|e_f(0) - e_f(1)| = 0$ and hence K_5 is divisor cordial. For the graph K_6, the labeling is given as follows.

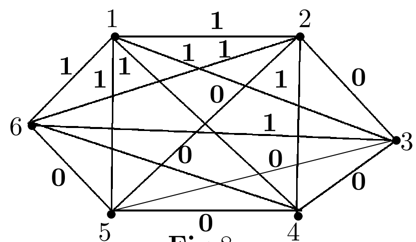

Fig.8

Here $|e_f(0) - e_f(1)| = 1$ and hence K_6 is divisor cordial. But K_n is not divisor cordial for $n \geq 7$, which will be proved in the following result.

Theorem 2.13 *K_n is not divisor cordial for $n \geq 7$.*

Proof If possible, let there be a divisor cordial labeling f for K_n. Let $v_1, \ldots, v_n$ be the vertices of K_n with $f(v_i) = i$. First we consider v_n. It contributes $d(n)$ and $(n-1) - d(n)$ respectively to $e_f(1)$ and $e_f(0)$. Consequently, the contribution of v_{n-1} to $e_f(1)$ and $e_f(0)$ are $d(n-1)$ and $n - 2 - d(n-1)$.

Proceeding likewise, we see that v_i contributes $d(i)$ and $i - 1 - d(i)$ to $e_f(1)$ and $e_f(0)$ respectively, for $i = n, n-1, \ldots, 2$. Then using Result 1.5, it follows that

$$\begin{aligned} |e_f(0) - e_f(1)| &= 2\{d(n) + \ldots + d(2)\} - \{(n-1) + \ldots + 1\} \\ &= 2\{D(n) - d(1)\} - \{\frac{(n-1)(n-2)}{2}\} \\ &= 2\{n \log n + n(2n-1) + \Delta(n) - 1\} - \{\frac{(n-1)(n-2)}{2}\} \\ &\geq 2 \end{aligned}$$

for $n \geq 7$. Thus, K_n is not divisor cordial. $\square$

Theorem 2.14 *$S(K_{1,n})$, the subdivision of the star $K_{1,n}$, is divisor cordial.*

Proof Let $V(S(K_{1,n})) = \{v, v_i, u_i : \ 1 \leq i \leq n\}$ and let $E(S(K_{1,n})) = \{vv_i, v_iu_i : \ 1 \leq i \leq$

$n\}$. Define f by

$$f(v) = 1,$$
$$f(v_i) = 2i(1 \le i \le n)$$
$$f(u_i) = 2i + 1 \quad (1 \le i \le n).$$

Here $e_f(0) = e_f(1) = n$. Hence $S(K_{1,n})$ is divisor cordial. □

The following example illustrates this theorem.

Example 1.15 Consider $S(K_{1,7})$.

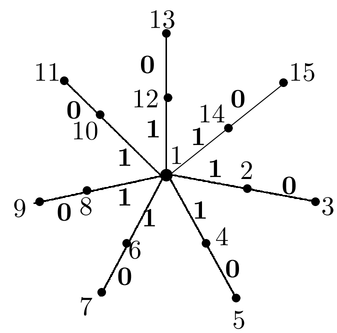

Fig.9

Here $e_f(0) = e_f(1) = 7$.

Theorem 2.16 *The bistar $B_{m,n}$ $(m \le n)$ is divisor cordial.*

Proof Let $V(B_{m,n}) = \{u, v, u_i, v_j : 1 \le i \le m, 1 \le j \le n\}$ and $E(B_{m,n}) = \{uu_i, vv_j : 1 \le i \le m, 1 \le j \le n\}$.

Case 1 $m = n$.

Define f by

$$f(u) = 2,$$
$$f(u_i) = 2i + 1, (1 \le i \le n)$$
$$f(v) = 1,$$
$$f(v_j) = 2i + 2(1 \le i \le n).$$

Since $e_f(0) = e_f(1) = n$, it follows that f gives a divisor cordial labeling.

Case 2 $m > n$.

Subcase 1 $m + n$ is even.

Define f by

$$f(u) = 2,$$
$$f(u_i) = 2i+1, (1 \le i \le \frac{m+n}{2}),$$
$$f(u_{\frac{m+n}{2}+i}) = 2n+2+2i, (1 \le i \le \frac{m-n}{2}),$$
$$f(v) = 1,$$
$$f(v_j) = 2j+2, 1 \le j \le n.$$

Since $e_f(0) = e_f(1) = \frac{m+n}{2}$, it follows that f is a divisor cordial labeling.

Subcase 2 $m+n$ is odd.

Define f by

$$f(u) = 2,$$
$$f(u_i) = 2i+1, (1 \le i \le \frac{m+n+1}{2}),$$
$$f(u_{\frac{m+n+1}{2}+i}) = 2n+2+2i, (1 \le i \le \frac{m-n-1}{2}),$$
$$f(v) = 1$$
$$f(v_j) = 2j+2, (1 \le j \le n)$$

Since $e_f(0) = \frac{m+n+1}{2}$ and $e_f(1) = \frac{m+n-1}{2}$, $|e_f(0) - e_f(1)| = 1$. It follows that f is a divisor cordial labeling. □

The Case ii of the Theorem 2.16 is illustrated in the following example.

Example 2.17 (1) Consider $B_{10,6}$.

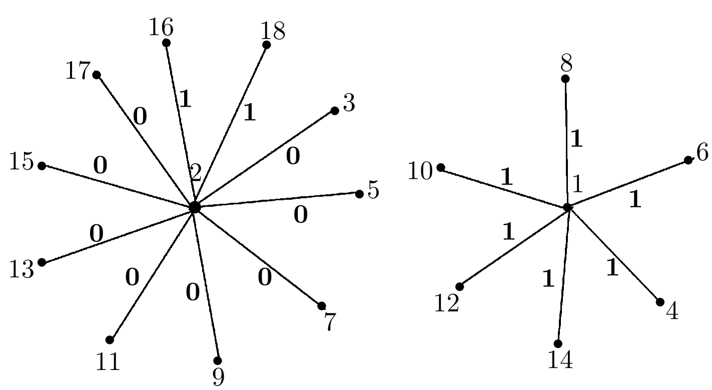

Fig.10

Here $e_f(0) = e_f(1) = 8$.

(2) Consider $B_{11,6}$. Here $e_f(0) = 9$, $e_f(1) = 8$.

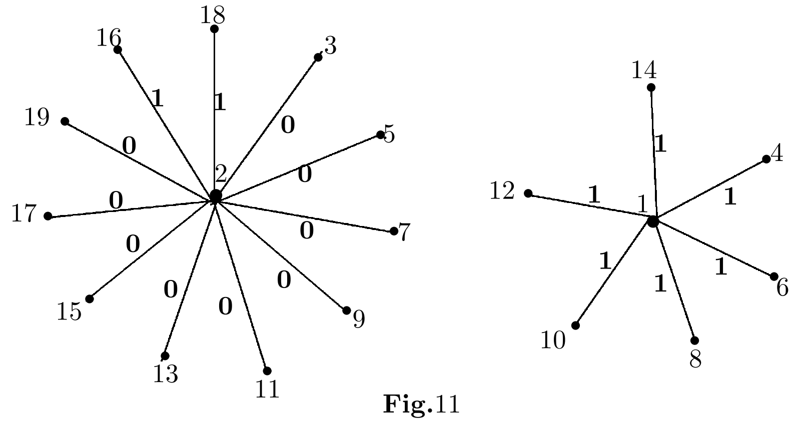

Fig.11

Theorem 2.18 *Let G be any divisor cordial graph of even size. Then the graph $G * K_{1,n}$ obtained by identifying the central vertex of $K_{1,n}$ with that labeled 2 in G is also divisor cordial.*

Proof Let q be the even size of G and let f be a divisor cordial labeling of G. Then it follows that, $e_f(0) = q/2 = e_f(1)$.

Let $v_1, v_2, \ldots, v_n$ be the pendant vertices of $K_{1,n}$. Extend f to $G * K_{1,n}$ by assigning $f(v_i) = |V| + i (1 \leq i \leq n)$. In $G * K_{1,n}$, we see that $|e_f(0) - e_f(1)| = 0$ or 1 according to n is even or odd. Thus, $G * K_{1,n}$ is also divisor cordial. □

Theorem 2.19 *Let G be any divisor cordial graph odd size. If n is even, then the graph $G*K_{1,n}$ obtained by identifying the central vertex of $K_{1,n}$ with that labeled with* 2 *in G is also divisor cordial.*

Proof Let q be the odd size of G and let f be a divisor cordial labeling of G. Then it follows that, $e_f(0) = e_f(1) + 1$ or $e_f(1) = e_f(0) + 1$.

Let $v_1, v_2, \ldots, v_n$ be the pendant vertices of $K_{1,n}$. Extend f to $G * K_{1,n}$ by assigning $f(v_i) = |V| + i$ $(1 \leq i \leq n)$. Since n is even, the edges of $K_{1,n}$ contribute equal numbers to both $e_f(1)$ and $e_f(0)$ in $G * K_{1,n}$. Thus, $G * K_{1,n}$ is divisor cordial. □

§3. Conclusion

In the subsequent papers, we will prove that some cycle related graphs such as dragon, corona, wheel, wheel with two centres, fan, double fan, shell, books and one point union of cycles are divisor cordial. Also we will prove some special classes of graphs such as full binary trees, some star related graphs, $G * K_{2,n}$ and $G * K_{3,n}$ are also divisor cordial.

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Complete Fuzzy Graphs

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Abstract: In this paper, we provide three new operations on fuzzy graphs; namely direct product, semi-strong product and strong product.We give sufficient conditions for each one of them to be complete and we show that if any of these products is complete, then at least one factor is a complete fuzzy graph. Moreover, we introduce and study the notion of balanced fuzzzy graph and give necessary and sufficient conditions for the preceding products of two fuzzy balanced graphs to be balanced and we prove that any isomorphic fuzzy graph to a balanced fuzzy graph must be balanced.

Key Words: Neutrosophic set, fuzzy graph, complete fuzzy graph, balanced fuzzy graph.

AMS(2010): 05C72

§1. Introduction

Graph theory has several interesting applications in system analysis, operations research and economics. Since most of the time the aspects of graph problems are uncertain, it is nice to deal with these aspects via the methods of fuzzy logic. The concept of fuzzy relation which has a widespread application in pattern recognition was introduced by Zadeh [8] in his landmark paper "Fuzzy sets" in 1965. Fuzzy graph and several fuzzy analogs of graph theoretic concepts were first introduced by Rosenfeld [6] in 1975. Sense then, fuzzy graph theory is finding an increasing number of applications in modelling real time systems where the level of information inherent in the system varies with different levels of precision. Fuzzy models are becoming useful because of their aim in reducing the differences between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems.

Mordeson and Peng [2] defined the concept of complement of fuzzy graph and studied some operations on fuzzy graphs. In [7], the definition of complement of a fuzzy graph was modified so that the complement of the complement is the original fuzzy graph, which agrees with the crisp graph case. Moreover some properties of self-complementary fuzzy graphs (fuzzy graphs that are isomorphic to their complements) and the complement of the operations of union, join and composition of fuzzy graphs that were introduced in [2] were studied. For more on the previous notions and the following ones, one can see [2]-[7].

A *neutrosophic set* based on neutrosophy, is defined for an element $x(T, I, F)$ belongs to

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the set if it is t true in the set, i indeterminate in the set, and f false, where t, i and f are real numbers taken from the sets T, I and F with no restriction on T, I, F nor on their sum $n = t + i + f$. Particularly, if $I = \emptyset$, we get the fuzzy set. Formally, a fuzzy subset of a non-empty set V is a mapping $\sigma : V \to [0,1]$ and a fuzzy relation μ on a fuzzy subset σ, is a fuzzy subset of $V \times V$. All throughout this paper, we assume that σ is reflexive, μ is symmetric and V is finite.

Definition 1.1([6]) *A fuzzy graph with V as the underlying set is a pair $G : (\sigma, \mu)$ where $\sigma : V \to [0,1]$ is a fuzzy subset and $\mu : V \times V \to [0,1]$ is a fuzzy relation on σ such that $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$, where $\wedge$ stands for minimum. The underlying crisp graph of G is denoted by $G^* : (\sigma^*, \mu^*)$ where $\sigma^* = \sup p(\sigma) = \{x \in V : \sigma(x) > 0\}$ and $\mu^* = \sup p(\mu) = \{(x,y) \in V \times V : \mu(x,y) > 0\}.H = (\sigma', \mu')$ is a* fuzzy subgraph *of G if there exists $X \subseteq V$ such that, $\sigma' : X \to [0,1]$ is a fuzzy subset and $\mu' : X \times X \to [0,1]$ is a fuzzy relation on σ' such that $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in X$.*

Definition 1.2([5]) *A fuzzy graph $G : (\sigma, \mu)$ is* complete *if $\mu(x,y) = \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$.*

Next, we recall the following two results from [7].

Lemma 1.3 *Let $G : (\sigma, \mu)$ be a self-complementary fuzzy graph. Then $\sum_{x,y \in V} \mu(x,y) = (1/2) \sum_{x,y \in V} (\sigma(x) \wedge \sigma(y))$.*

Lemma 1.4 *Let $G : (\sigma, \mu)$ be a fuzzy graph with $\mu(x,y) = (1/2)(\sigma(x) \wedge \sigma(y))$ for all $x, y \in V$.Then G is self-complementary.*

Definition 1.5([1]) *Two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with crisp graph $G_2^* : (V_2, E_2)$ are* isomorphic *if there exists a bijection $h : V_1 \to V_2$ such that $\sigma_1(x) = \sigma_2(h(x))$ and $\mu_1(x,y) = \mu_2(h(x), h(y))$ for all $x, y \in V_1$.*

Lemma 1.6([3]) *Any two isomorphic fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ satisfy $\sum_{x \in V_1} \sigma_1(x) = \sum_{x \in V_2} \sigma_2(x)$ and $\sum_{x,y \in V_1} \mu_1(x,y) = \sum_{x,y \in V_2} \mu_2(x,y)$.*

In this paper, we provide three new operations on fuzzy graphs, namely direct product, semi-strong product and strong product. We give sufficient conditions for each one of them to be complete and show that if any one of these product of two fuzzy graphs is complete, then at least one of the two fuzzy graphs must be complete. Moreover, we introduce and study the notion of balanced fuzzy graph and show that this notion is weaker than complete and we give necessary and sufficient conditions for the direct product, semi-strong product and strong product of two balanced fuzzy graphs to be balanced. Finally we prove that given a balanced fuzzy graph G, then any isomorphic fuzzy graph to G must be balanced.

§2. Complete Fuzzy Graphs

We begin this section by defining three new fuzzy graphs products.

Definition 2.1 *The direct product of two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with crisp graph $G_2^* : (V_2, E_2)$,where we assume that $V_1 \cap V_2 = \varnothing$, is defined to be the fuzzy graph $G_1 \sqcap G_2 : (\sigma_1 \sqcap \sigma_2, \mu_1 \sqcap \mu_2)$ with crisp graph $G^* : (V_1 \times V_2, E)$ where*

$$E = \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\},$$

$$(\sigma_1 \sqcap \sigma_2)(u, v) = \sigma_1(u) \wedge \sigma_2(v), \text{ for all } (u, v) \in V_1 \times V_2 \text{ and}$$
$$(\mu_1 \sqcap \mu_2)((u_1, v_1)(u_2, v_2)) = \mu_1(u_1, u_2) \wedge \mu_2(v_1, v_2).$$

Definition 2.2 *The semi-strong product of two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with crisp graph $G_2^* : (V_2, E_2)$,where we assume that $V_1 \cap V_2 = \varnothing$, is defined to be the fuzzy graph $G_1 \cdot G_2 : (\sigma_1 \cdot \sigma_2, \mu_1.\mu_2)$ with crisp graph $G^* : (V_1 \times V_2, E)$ where*

$$E = \{(u, v_1)(u, v_2) : u \in V_1, (v_1, v_2) \in E_2\} \cup \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\},$$

$$(\sigma_1 \cdot \sigma_2)(u, v) = \sigma_1(u) \wedge \sigma_2(v), \text{ for all } (u, v) \in V_1 \times V_2,$$
$$(\mu_1.\mu_2)((u, v_1)(u, v_2)) = \sigma_1(u) \wedge \mu_2(v_1, v_2) \quad \text{and}$$
$$(\mu_1.\mu_2)((u_1, v_1)(u_2, v_2)) = \mu_1(u_1, u_2) \wedge \mu_2(v_1, v_2).$$

Definition 2.3 *The strong product of two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with crisp graph $G_2^* : (V_2, E_2)$, where we assume that $V_1 \cap V_2 = \varnothing$, is defined to be the fuzzy graph $G_1 \otimes G_2 : (\sigma_1 \otimes \sigma_2, \mu \otimes \mu_2)$ with crisp graph $G^* : (V_1 \times V_2, E)$ where*

$$E = \{(u, v_1)(u, v_2) : u \in V_1, (v_1, v_2) \in E_2\} \cup \{(u_1, w)(u_2, w) : w \in V_2, (u_1, u_2) \in E_1\} \cup \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\},$$

$$(\sigma_1 \otimes \sigma_2)(u, v) = \sigma_1(u) \wedge \sigma_2(v), \text{ for all } (u, v) \in V_1 \times V_2,$$
$$(\mu_1 \otimes \mu_2)((u, v_1)(u, v_2)) = \sigma_1(u) \wedge \mu_2(v_1, v_2) \ ,$$
$$(\mu_1 \otimes \mu_2)((u_1, w)(u_2, w)) = \sigma_2(w) \wedge \mu_1(u_1, u_2) \text{ and}$$
$$(\mu_1 \otimes \mu_2)((u_1, v_1)(u_2, v_2)) = \mu_1(u_1, u_2) \wedge \mu_2(v_1, v_2).$$

Next, we show that the direct product, the semi-strong product and the strong product of two complete fuzzy graphs are again fuzzy complete graphs.

Theorem 2.4 *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are complete fuzzy graphs, then $G_1 \sqcap G_2$ is complete.*

Proof If $(u_1, v_1)(u_2, v_2) \in E$,then since G_1 and G_2 are complete

$$\begin{aligned}(\mu_1 \sqcap \mu_2)((u_1, v_1)(u_2, v_2)) &= \mu_1(u_1, u_2) \wedge \mu_2(v_1, v_2)\\ &= \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)\\ &= (\sigma_1 \sqcap \sigma_2)((u_1, v_1)) \wedge (\sigma_1 \sqcap \sigma_2)((u_2, v_2)).\end{aligned}$$

Hence, $G_1 \sqcap G_2$ is complete. $\square$

Theorem 2.5 *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are complete fuzzy graphs, then $G_1 \bullet G_2$ is complete.*

Proof If $(u, v_1)(u, v_2) \in E$, then

$$\begin{aligned}(\mu_1 \bullet \mu_2)((u, v_1)(u, v_2)) &= \sigma_1(u) \wedge \mu_2(v_1, v_2) \\ &= \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2) \text{ (since } G_2 \text{ is complete)} \\ &= (\sigma_1 \bullet \sigma_2)((u, v_1)) \wedge (\sigma_1 \bullet \sigma_2)((u, v_2)).\end{aligned}$$

If $(u_1, v_1)(u_2, v_2) \in E$, then since G_1 and G_2 are complete

$$\begin{aligned}(\mu_1 \bullet \mu_2)((u_1, v_1)(u_2, v_2)) &= \mu_1(u_1, u_2) \wedge \mu_2(v_1, v_2) \\ &= \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2) \\ &= (\sigma_1 \bullet \sigma_2)((u_1, v_1)) \wedge (\sigma_1 \bullet \sigma_2)((u_2, v_2)).\end{aligned}$$

Hence, $G_1 \bullet G_2$ is complete. □

Theorem 2.6 *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are complete fuzzy graphs, then $G_1 \otimes G_2$ is complete.*

Proof If $(u, v_1)(u, v_2) \in E$, then

$$\begin{aligned}(\mu_1 \otimes \mu_2)((u, v_1)(u, v_2)) &= \sigma_1(u) \wedge \mu_2(v_1, v_2) \\ &= \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2) \text{ (since } G_2 \text{ is complete)} \\ &= (\sigma_1 \otimes \sigma_2)((u, v_1)) \wedge (\sigma_1 \otimes \sigma_2)((u, v_2)).\end{aligned}$$

If $(u_1, w)(u_2, w) \in E$, then

$$\begin{aligned}(\mu_1 \otimes \mu_2)((u_1, w)(u_2, w)) &= \sigma_2(w) \wedge \mu_1(u_1, u_2) \\ &= \sigma_2(w) \wedge \sigma_1(u_1) \wedge \sigma_1(u_2) \text{ (since} G_1 \text{ is complete)} \\ &= (\sigma_1 \otimes \sigma_2)((u_1, w)) \wedge (\sigma_1 \otimes \sigma_2)((u_2, w)).\end{aligned}$$

If $(u_1, v_1)(u_2, v_2) \in E$, then since G_1 and G_2 are complete

$$\begin{aligned}(\mu_1 \otimes \mu_2)((u_1, v_1)(u_2, v_2)) &= \mu_1(u_1, u_2) \wedge \mu_2(v_1, v_2) \\ &= \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2) \\ &= (\sigma_1 \otimes \sigma_2)((u_1, v_1)) \wedge (\sigma_1 \otimes \sigma_2)((u_2, v_2)).\end{aligned}$$

Hence, $G_1 \otimes G_2$ is complete. □

An interesting property of complement is given next.

Theorem 2.7 *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are complete fuzzy graphs, then $\overline{G_1 \otimes G_2} \simeq \overline{G_1} \otimes \overline{G_2}$.*

Proof Let $G : (\sigma, \overline{\mu}) = \overline{G_1 \otimes G_2}$, $\overline{\mu} = \overline{\mu_1 \otimes \mu_2}$, $\overline{G^*} = (V, \overline{E})$, $\overline{G_1} : (\sigma_1, \overline{\mu_1})$, $\overline{G_1^*} = (V_1, \overline{E_1})$, $\overline{G_2} : (\sigma_2, \overline{\mu_2})$, $\overline{G_2^*} = (V_2, \overline{E_2})$and $\overline{G_1} \otimes \overline{G_2} : (\sigma_1 \otimes \sigma_2, \overline{\mu_1} \otimes \overline{\mu_2})$. We only need to show $\overline{\mu_1 \otimes \mu_2} = \overline{\mu_1} \otimes \overline{\mu_2}$. For any arc e joining nodes of V, we have the following cases:

Case 1 $e = (u, v_1)(u, v_2)$ where $(v_1, v_2) \in E_2$.Then as G is complete by Theorem 2.6, $\overline{\mu}(e) = 0$. On the other hand, $(\overline{\mu_1} \otimes \overline{\mu_2})(e) = 0$ since $(v_1, v_2) \notin \overline{E_2}$.

Case 2 $e = (u, v_1)(u, v_2)$ where $(v_1, v_2) \in E_2$ and $v_1 \neq v_2$. Since $e \in E$, $\mu(e) = 0$ and $\overline{\mu}(e) = \sigma(u, v_1) \wedge \sigma(u, v_2) = \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$ and as $(v_1, v_2) \in \overline{E_2}$, $(\overline{\mu_1} \otimes \overline{\mu_2})(e) = \sigma_1(u) \wedge \mu_2(v_1, v_2)$ and as $\overline{G_2}$ is complete, $(\overline{\mu_1} \otimes \overline{\mu_2})(e) = \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$.

Case 3 $e = (u_1, w)(u_2, w)$ where $(u_1, u_2) \in E_1$. Since $e \in E$, $\overline{\mu}(e) = 0$ and as $(u_1, u_2) \notin \overline{E_1}$, $(\overline{\mu_1} \otimes \overline{\mu_2})(e) = 0$.

Case 4 $e = (u_1, w)(u_2, w)$ where $(u_1, u_2) \notin E_1$. Since $e \notin E$, $\mu(e) = 0$ and $\overline{\mu}(e) = \sigma(u_1, w) \wedge \sigma(u_2, w) = \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(w)$ and as $(u_1, u_2) \in \overline{E_1}$, $(\overline{\mu_1} \otimes \overline{\mu_2})(e) = \sigma_2(w) \wedge \overline{\mu_1}(u_1, u_2) = \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(w)$ since $\overline{G_1}$ is complete.

Case 5 $e = (u_1, v_1)(u_2, v_2)$ where $(u_1, u_2) \notin E_1$ and $v_1 \neq v_2$. Since $e \in E$, $\overline{\mu}(e) = 0$ and as $(u_1, u_2) \in \overline{E_1}$, $(\overline{\mu_1} \otimes \overline{\mu_2})(e) = 0$.

Case 6 $e = (u_1, v_1)(u_2, v_2)$ where $(u_1, u_2) \in E_1$ and $v_1 \neq v_2$. Since $e \notin E$, $\overline{\mu}(e) = 0$ and so $\overline{\mu}(e) = \sigma(u_1, w) \wedge \sigma(u_2, w) = \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$ and as $(u_1, u_2) \in \overline{E_1}$ and as $\overline{G_1}$ is complete, $(\overline{\mu_1} \otimes \overline{\mu_2})(e) = \overline{\mu_1}(u_1, u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2) = \overline{\mu}(e)$.

Case 7 $e = (u_1, v_1)(u_2, v_2)$ where $(u_1, u_2) \notin E_1$ and $(v_1, v_2) \notin E_2$. Since $e \notin E$, $\mu(e) = 0$ and $\overline{\mu}(e) = \sigma(u_1, w) \wedge \sigma(u_2, w) = \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$. As $(u_1, u_2) \in \overline{E_1}$ and if $v_1 = v_{2,}$ then we have Case 4. If $(u_1, u_2) \in \overline{E_1}$ and $v_1 \neq v_{2,}$ then we have Case 6.

In all cases $\overline{\mu_1 \otimes \mu_2} = \overline{\mu_1} \otimes \overline{\mu_2}$ and therefore, $\overline{G_1 \otimes G_2} \simeq \overline{G_1} \otimes \overline{G_2}$. □

By similar arguments to those in the preceding result and using Theorems 2.4 and 2.5, we can prove the following result.

Theorem 2.8 *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$are fuzzy complete graphs, then $\overline{G_1 \sqcap G_2} \simeq \overline{G_1} \sqcap \overline{G_2}$ and $\overline{G_1 \bullet G_2} \simeq \overline{G_1} \bullet \overline{G_2}$.*

Next, we show that if the direct product, the semi-strong product or the strong product of two fuzzy graphs is complete, then at least one of the two fuzzy graphs must be complete.

Theorem 2.9 *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are fuzzy graphs such that $G_1 \sqcap G_2$ is complete, then at least G_1 or G_2 must be complete.*

Proof Suppose that G_1 and G_2 are not complete. Then there exists at least one $(u_1, v_1) \in E_1$ and $(u_2, v_2) \in E_2$ such that

$$\begin{aligned} \mu_1((u_1, v_1)) &< \sigma_1(u_1) \wedge \sigma_1(v_1)) \text{ and} \\ \mu_2((u_2, v_2)) &< \sigma_2(u_2) \wedge \sigma_2(v_2)) \end{aligned}$$

Now

$$\begin{aligned} (\mu_1 \sqcap \mu_2)((u_1, v_1)(u_2, v_2)) &= \mu_1(u_{1,}u_2) \wedge \mu_2(v_1, v_2) \\ &< \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2) \text{ (since } G_1 \text{ and } G_2 \text{ are complete).} \end{aligned}$$

But $(\sigma_1 \sqcap \sigma_2)((u_1, v_1)) = \sigma_1(u_1) \wedge \sigma_2(v_1)$ and $(\sigma_1 \sqcap \sigma_2)((u_2, v_2)) = \sigma_1(u_2) \wedge \sigma_2(v_2)$.Thus

$$\begin{aligned}(\sigma_1 \sqcap \sigma_2)((u_1, v_1)) \wedge (\sigma_1 \sqcap \sigma_2)((u_2, v_2)) &= \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2) \\ &> (\mu_1 \sqcap \mu_2)((u_1, v_1)(u_2, v_2)).\end{aligned}$$

Hence, $G_1 \sqcap G_1$ is not complete, a contradiction. □

The next result can be proved in a similar manor to the preceding one.

Theorem 2.10 *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are fuzzy graphs such that $G_1 \bullet G_2$ or $G_1 \otimes G_2$ is complete, then at least G_1 or G_2 must be complete.*

§3. Balanced Fuzzy Graphs

We begin this section by defining the density of a fuzzy graph and balanced fuzzy graphs. We then show that any complete fuzzy graph is balanced, but the converse need not be true.

Definition 3.1 *The density of a fuzzy graph $G : (\sigma, \mu)$ is*

$$D(G) = 2(\sum_{u,v \in V} \mu(u, v))/(\sum_{u,v \in V} (\sigma(u) \wedge \sigma(v))).$$

G is balanced *if $D(H) \leq D(G)$ for all fuzzy non-empty subgraphs H of G.*

Theorem 3.2 *Any complete fuzzy graph is balanced.*

Proof Let G be a complete fuzzy graph. Then

$$\begin{aligned} D(G) &= 2(\sum_{u,v \in V} \mu(u, v))/(\sum_{u,v \in V} (\sigma(u) \wedge \sigma(v))) \\ &= 2(\sum_{u,v \in V} (\sigma(u) \wedge \sigma(v)))/(\sum_{u,v \in V} \mu(u, v)) = 2 \end{aligned}$$

.If H is a non-empty fuzzy subgraph of G,then

$$\begin{aligned} D(H) &= 2(\sum_{u,v \in V(H)} \mu(u, v))/(\sum_{u,v \in V(H)} (\sigma(u) \wedge \sigma(v))) \\ &\leq 2(\sum_{u,v \in V(H)} (\sigma(u) \wedge \sigma(v))/(\sum_{u,v \in V(H)} (\sigma(u) \wedge \sigma(v))) \\ & \quad 2(\sum_{u,v \in V} (\sigma(u) \wedge \sigma(v))/(\sum_{u,v \in V} (\sigma(u) \wedge \sigma(v))) \\ &= 2 = D(G). \end{aligned}$$

Thus G is balanced. □

The converse of the preceding result need not be true.

Example 3.3 The following fuzzy graph $G : (\sigma, \mu)$ is a balanced graph that is not complete.

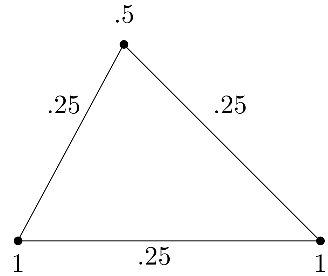

Fig.1

We next provide two types of fuzzy graphs each with density equals 1.

Theorem 3.4 *Every self-complementary fuzzy graph has density equals 1.*

Proof Let G be a self-complementary fuzzy graph. Then by Lemma 1.3,

$$\begin{aligned} D(G) &= 2(\sum_{u,v\in V} \mu(u,v))/(\sum_{u,v\in V} (\sigma(u) \wedge \sigma(v))) \\ &= 2(\sum_{u,v\in V} \mu(u,v))/(2\sum_{u,v\in V} \mu(u,v))) = 1. \end{aligned}$$

This completes the proof. □

The converse of the preceding result need not be true.

Example 3.5 The following fuzzy graph $G : (\sigma, \mu)$ has density equals 1, but it is not self-complementary.

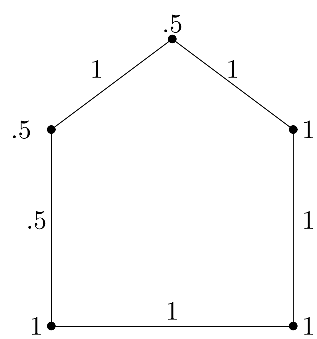

Fig.2

Theorem 3.6 *Let $G : (\sigma, \mu)$ be a fuzzy graph such that $\mu(u,v) = (1/2)(\sigma(u) \wedge \sigma(v))$, for all $u, v \in V$. Then $D(G) = 1$.*

Proof Let $G : (\sigma, \mu)$ be a fuzzy graph such that $\mu(u,v) = (1/2)(\sigma(u) \wedge \sigma(v))$, for all $u, v \in V$. By Lemma 1.4, G is self-complementary and thus by the preceding Theorem $D(G) = 1$. □

Next, we prove the following lemma that we use to give necessary and sufficient conditions for the direct product, semi-strong product and strong product of two fuzzy balanced graphs to be balanced.

Lemma 3.7 *Let G_1 and G_2 be fuzzy graphs. Then $D(G_i) \leq D(G_1 \sqcap G_2)$ for $i = 1, 2$ if and only if $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$.*

Proof If $D(G_i) \leq D(G_1 \sqcap G_2)$ for $i = 1, 2$, then

$$\begin{aligned}
D(G_1) &= 2(\sum_{u_1,u_2\in V_1} \mu_1(u_1,u_2))/(\sum_{u_1,u_2\in V_1} (\sigma_1(u_1) \wedge \sigma_1(u_2))) \\
&\geq 2(\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} \mu_1(u_1,u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2))/(\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} (\sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2))) \\
&= 2(\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} \mu_1(u_1,u_2) \wedge \mu_2(v_1,v_2))/(\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} (\sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2))) \\
&= 2(\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} \mu_1 \sqcap \mu_2((u_1,u_2)(v_1,v_2)))/(\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} (\sigma_1 \sqcap \sigma_2((u_1,u_2)(v_1,v_2))) \\
&= D(G_1 \sqcap G_2).
\end{aligned}$$

Hence $D(G_1) \geq D(G_1 \sqcap G_2)$ and thus $D(G_1) = D(G_1 \sqcap G_2)$. Similarly, $D(G_2) = D(G_1 \sqcap G_2)$. Therefore, $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$. □

Theorem 3.8 *Let G_1 and G_2 be fuzzy balanced graphs. Then $G_1 \sqcap G_2$ is balanced if and only if $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$.*

Proof If $G_1 \sqcap G_2$ is balanced, then $D(G_i) \leq D(G_1 \sqcap G_2)$ for $i = 1, 2$ and by Lemma 3.7, $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$.

Conversely, if $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$ and H is a fuzzy subgraph of $G_1 \sqcap G_2$, then there exist fuzzy subgraphs H_1 of G_1 and H_2 of G_2. As G_1 and G_2 are balanced and $D(G_1) = D(G_2) = n_1/r_1$, then $D(H_1) = a_1/b_1 \leq n_1/r_1$ and $D(H_2) = a_2/b_2 \leq n_1/r_1$. Thus $a_1r_1 + a_2r_1 \leq b_1n_1 + b_2n_1$ and hence $D(H) \leq (a_1 + a_2)/(b_1 + b_2) \leq n_1/r_1 = D(G_1 \sqcap G_2)$. Therefore, $G_1 \sqcap G_2$ is balanced. □

By similar arguments to those in Lemma 3.7 and Theorem 3.8, we can prove the following result:

Theorem 3.9 *Let G_1 and G_2 be fuzzy balanced graphs. Then*

(1) *$G_1 \bullet G_2$ is balanced if and only if $D(G_1) = D(G_2) = D(G_1 \bullet G_2)$.*
(2) *$G_1 \otimes G_2$ is balanced if and only if $D(G_1) = D(G_2) = D(G_1 \otimes G_2)$.*

We end this section by showing that isomorphism between fuzzy graphs preserve balanced.

Theorem 3.10 *Let G_1 and G_2 be isomorphic fuzzy graphs. If G_2 is balanced, then G_1 is balanced.*

Proof Let $h : V_1 \to V_2$ be a bijection such that $\sigma_1(x) = \sigma_2(h(x))$ and $\mu_1(x,y) = \mu_2(h(x),h(y))$ for all $x,y \in V_1$. By Lemma 1.6, $\sum_{x\in V_1}\sigma_1(x) = \sum_{x\in V_2}\sigma_2(x)$ and $\sum_{x,y\in V_1}\mu_1(x,y) = \sum_{x,y\in V_2}\mu_2(x,y)$. If $H_1 = (\sigma_1^{'}, \mu_1^{'})$ is a fuzzy subgraph of G_1 with underlying set W, then $H_2 = (\sigma_2^{'}, \mu_2^{'})$ is a fuzzy subgraph of G_2 with underlying set $h(W)$ where $\sigma_2^{'}(h(x)) = \sigma_1^{'}(x)$ and $\mu_2^{'}(h(x),h(y)) = \mu_1^{'}(x,y)$ for all $x,y \in W$. Since G_2 is balanced, $D(H_2) \le D(G_2)$ and so

$$2(\sum_{x,y\in W}\mu_2^{'}(h(x),h(y)))/(\sum_{x,y\in W}(\sigma_2^{'}(x)\wedge\sigma_2^{'}(y))) \le 2(\sum_{x,y\in V_2}\mu_2(x,y))/(\sum_{x,y\in V_2}(\sigma_2(x)\wedge\sigma_2(y)))$$

and so

$$\begin{aligned} 2(\sum_{x,y\in W}\mu_1(x,y))/(\sum_{x,y\in W}(\sigma_2^{'}(x)\wedge\sigma_2^{'}(y))) &\le 2(\sum_{x,y\in V_1}\mu_1(x,y))/(\sum_{x,y\in V_2}(\sigma_2(x)\wedge\sigma_2(y))) \\ &\le D(G_1). \end{aligned}$$

Therefore, G_1 is balanced. □

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A Variation of Decomposition Under a Length Constraint

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Abstract: Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be graphical properties. A *Smarandachely* $(\mathscr{P}_1, \mathscr{P}_2)$-*decomposition* of a graph G is a decomposition of G into subgraphs $G_1, G_2, \cdots, G_l \in \mathscr{P}$ such that $G_i \in \mathscr{P}_1$ or $G_i \notin \mathscr{P}_2$ for integers $1 \leq i \leq l$. Particularly, if $\mathscr{P}_2 = \emptyset$, i.e., a usual *decomposition* of a graph, is a collection of its subgraphs whose union equals the edge set of the graph. In this paper we introduce and initiate a study of a new variation of decomposition namely *equiparity induced path decomposition* of a graph which is defined to be a decomposition in which all the members are induced paths having same parity.

Key Words: Smarandachely $(\mathscr{P}_1, \mathscr{P}_2)$-decomposition, induced path decomposition, equiparity path decomposition, equiparity induced path decomposition.

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§1. Introduction

By a graph $G = (V, E)$ we mean a non-trivial, finite, connected and undirected graph without loops or multiple edges. For terms not defined here, we refer to [3]. Throughout the paper the order and size of G are denoted by n and m respectively.

The origin of the study of graph decomposition and factorization can be seen in various combinatorial problems most of which emerged in the 19th century. Among them the best known are Kirkman's problem of 15 strolling school girls, Dudney's problem of handcuffed prisoners, Euler's problem of 36 army officers , Kirkman's problem of knights and Lucas dancing round problem. However, the earliest works in this direction are not explicitly related to graph decompositions. The first papers (due to J.Peterson, A.B.Kempe, P.G.Tait, P.J.Heawood, D.Konig and others) appeared soon afterwards at the turn of the 19th century. Since that time the interest in graph decompositions has been on increase and real upsurge is witnessed after 1950. Nowadays, graph decomposition problems rank among the most prominent areas of research in graph theory and combinatorics.

As we know a *decomposition* of G is a collection $\psi = \{H_1, H_2, H_3, \ldots, H_k\}$ of subgraphs of G such that every edge of G belongs to exactly one H_i. If each H_i is a path in G, then ψ is called

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a *path decomposition* of G. The minimum cardinality of a path decomposition of G is called the path decomposition number and is denoted by $\pi_a(G)$. The concept of path decomposition was introduced by Harary [4] in the year 1970 and was further studied by Schwenk, Peroche, Stanton, Cowan and James ([5], [7], [9]). Following Harary several variations of decomposition have been introduced and extensively studied by imposing conditions on the members of the decomposition. For instance, unrestricted path cover [5], geodesic path partition [10], simple path cover [2], induced path decomposition [8], equiparity path decomposition [6], graphoidal cover [1] are some variations of decomposition. In this direction we introduce the concept of *equiparity induced path decomposition* and initiate a study of this new decomposition.

§2. Equiparity Induced Path Decomposition

In this section we define the equiparity induced path decomposition and the parameter equiparity induced path decomposition number of a graph G and determine this parameter for some standard graphs such as complete multipartite graphs, wheels, fans, double fans and generalized Petersen graphs. Further we explore the relation between this parameter and some of the existing path decomposition parameters of a given graph G.

Definition 2.1 *An Equiparity induced path decomposition ($\mathcal{ED}$) of a graph G is a path decomposition ψ of G such that the elements of ψ are induced paths having same parity. That is, an $\mathcal{ED}$ is an equiparity as well as induced path decomposition of G. The minimum cardinality of an $\mathcal{ED}$ for a graph G is called the equiparity induced path decomposition number and is denoted by $\pi_{pi}(G)$. Any $\mathcal{ED}$ of G such that $|\psi| = \pi_{pi}(G)$ is called a* minimum equiparity induced path decomposition *of G.*

An equiparity induced path decomposition ψ of a graph is said to be an even parity induced path decomposition ($E\mathcal{ED}$) or an odd parity induced path decomposition ($O\mathcal{ED}$) according as all the paths in ψ are of even length or odd length.

Remark 2.2 Obviously, for any graph G, the edge set $E(G)$ itself is an $\mathcal{ED}$ so that every graph G admits an $\mathcal{ED}$ and hence the parameter π_{pi} is well defined for all graphs.

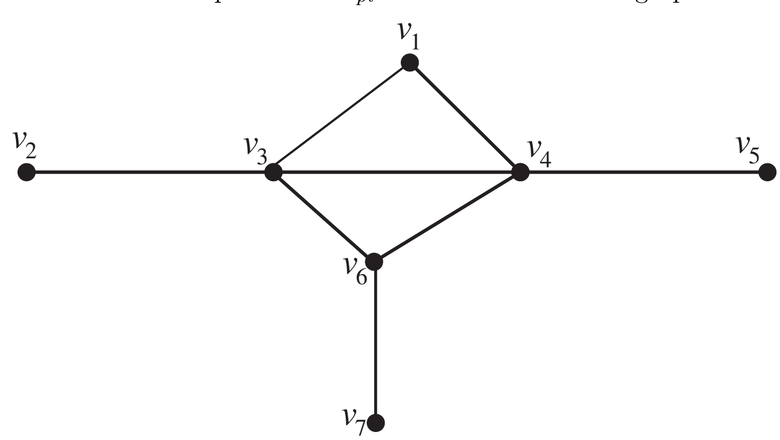

Fig.1

Example 2.3 (i) Consider the graph G given in Fig.1. Let

$$\psi_1 = \{(v_1, v_3, v_6, v_7), (v_2, v_3, v_4, v_5), (v_1, v_4), (v_4, v_6)\}$$
$$\psi_2 = \{(v_2, v_3, v_1), (v_3, v_6, v_7), (v_1, v_4, v_6), (v_3, v_4, v_5)\}$$
$$\psi_3 = \{(v_2, v_3, v_4, v_6), (v_7, v_6, v_3, v_1, v_4, v_5)\}$$
$$\psi_4 = \{(v_1, v_3, v_6, v_7), (v_1, v_4, v_6), (v_2, v_3, v_4, v_5)\}.$$

Then ψ_1 and ψ_2 are $\mathcal{ED}$s of G. Also, all the paths in ψ_1 are of odd length, where as in ψ_2 they are even. But ψ_3 and ψ_4 are not $\mathcal{ED}$s for G, because the former is not induced and the latter is not equiparity. We also note that the minimum cardinality of an $\mathcal{ED}$ for G is 4 and thus both ψ_1 and ψ_2 are minimum $\mathcal{ED}$s of G.

(ii) For paths, the value of π_{pi} is always 1.

(iii) If C_n denotes the cycle on n vertices, then

$$\pi_{pi}(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$$

(iv) Since the edges are the only induced paths in the complete graph K_n on n vertices, we have

$$\pi_{pi}(K_n) = \frac{n(n-1)}{2}.$$

Remark 2.4 If G is a graph of odd size, then it admits only an $O\mathcal{ED}$ and so the value of π_{pi} must be odd. However it is possible for a graph of even size to have both $O\mathcal{ED}$ and $E\mathcal{ED}$; in fact it can permit an $O\mathcal{ED}$ and an $E\mathcal{ED}$ of minimum cardinality as in Example 2.3(i). Also, the value of π_{pi} for a graph with even size can be both even or odd (for example see Theorem 2.9).

To determine the value of π_{pi} for a given graph, the following theorem is useful. If $P = (v_1, v_2, v_3, \ldots, v_n)$ is a path in a graph $G = (V, E)$, the vertices $v_2, v_3, \ldots, v_{n-1}$ are called *internal vertices* of P while v_1 and v_n are called *external vertices* of P.

Theorem 2.5 *For an $\mathcal{ED}$ ψ of a graph G, let $t_\psi = \sum_{p\in\psi} t(P)$ where $t(P)$ denotes the number of internal vertices of the path P and let $t = \max t_\psi$, where the maximum is taken over all $\mathcal{ED}, \psi$ of G. Then $\pi_{pi}(G) = m - t$.*

Proof Let ψ be any $\mathcal{ED}$ of G. Then

$$\begin{aligned} m &= \sum_{p\in\psi} |E(P)| = \sum_{p\in\psi} [t(P) + 1] \\ &= \{\sum_{p\in\psi} t(P)\} + |\psi| = t_\psi + |\psi| \end{aligned}$$

Hence $|\psi| = m - t_\psi$ so that $\pi_{pi} = m - t$. $\square$

The following corollaries are the immediate consequences of the above theorem.

Corollary 2.6 *If G is a graph with k vertices of odd degree, then*

$$\pi_{pi}(G) = \frac{k}{2} + \sum_{v\in V(G)} \lfloor \frac{\deg v}{2} \rfloor - t.$$

Corollary 2.7 *For any graph $G, \pi_{pi}(G) \geqslant \frac{k}{2}$. Further, equality holds if and only if there exists an equiparity induced path decomposition ψ of G such that every vertex v of G is an internal vertex of $\lfloor\frac{\deg v}{2}\rfloor$ paths in ψ.*

In the following results, we determine the value of π_{pi} for wheels, complete multipartite graphs, fans, double fans and the generalized Petersen graph.

Theorem 2.8 *If W_n denotes the wheel on n vertices, then*

$$\pi_{pi}(W_n) = \begin{cases} \frac{(n+3)}{2} & \textit{when } n \textit{ is odd} \\ (n+2) & \textit{when } n \textit{ is even} \end{cases}$$

Proof If $n = 4$, then $W_4 = K_4$ so that $\pi_{pi}(W_4) = 6$. Now let us assume that $n \geqslant 5$. Let $V(W_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $E(W_n) = \{v_n v_i : 1 \leqslant i \leqslant n-1\} \cup \{v_i v_{i+1} : 1 \leqslant i \leqslant n-2\} \cup \{v_{n-1}, v_1\}$.

Case 1 n is odd.

Let

$$\begin{aligned} &P_i = (v_i, v_n, v_{i+\frac{n-1}{2}}) \quad \text{for all } i = 1, 2, 3, \ldots, \frac{n-1}{2}, \\ &Q_1 = (v_1, v_2, v_3) \text{ and} \\ &Q_2 = (v_3, v_4, v_5, \ldots, v_{n-1}, v_1). \end{aligned}$$

Then, $\psi = \{P_1, P_2, P_3, \ldots, P_{\frac{n-1}{2}}, Q_1, Q_2\}$ is an $\mathcal{ED}$ of W_n so that $\pi_{pi}(W_n) \leqslant |\psi| = \frac{n-1}{2} + 2 = \frac{n+3}{2}$. Further, any induced path containing the vertex v_n is of length at most two and so the minimum number of induced paths required to decompose the spokes (the edges $v_1v_n, v_2v_n, \ldots, v_{n-1}v_n$) of the wheel is $\frac{n-1}{2}$. Also, since the outer cycle is of even length, we need at least two induced paths to decompose it and hence $\pi_{pi}(W_n) \geq \frac{n-1}{2} + 2 = \frac{n+3}{2}$ so that $\pi_{pi}(W_n) = \frac{n+3}{2}$ when n is odd.

Case 2 n is even.

Let

$$\begin{aligned} &P_i = (v_i, v_n) \quad \text{for all } i = 1, 2, 3, \ldots, n-1, \\ &Q_1 = (v_1, v_2, v_3, \ldots, v_{n-2}), \\ &Q_2 = (v_{n-2}, v_{n-1}) \text{ and} \\ &Q_3 = (v_{n-1}, v_1) \end{aligned}$$

Then $\psi = \{P_1, P_2, P_3, \ldots, P_{n-1}, Q_1, Q_2, Q_3\}$ is an $\mathcal{ED}$ so that $\pi_{pi}(W_n) \leqslant |\psi| = n+2$. Moreover, an induced path of W_n cannot contain both an edge of the outer cycle and a spoke. Now, the outer cycle can be decomposed into induced paths of odd length only, because the outer cycle

is odd. Therefore, we can have only an $O\mathcal{ED}$ and obviously that will consist of all the $n-1$ spokes together with at least three induced paths of odd length which decompose the outer cycle so that $|\psi| \geqslant n+2$ and this completes the proof of the theorem. □

Theorem 2.9 *If G is a complete k-partite graph $K_{m_1,m_2,m_3,\ldots,m_k}$, with m edges, then*

$$\pi_{pi}(G) = \begin{cases} \dfrac{m}{2} & \text{if } m_i \text{ is odd for at most one } i \\ m & \text{otherwise} \end{cases}$$

Proof Let $(V_1, V_2, \ldots, V_k)$ be the partition of $V(G)$. Obviously the induced paths in G are of length at most two and hence any $\mathcal{ED}$ of G consists of either single edges alone or induced paths of length 2. Moreover the end vertices of the induced paths of length two lie in the same partition. Therefore, when there exist two parts V_i and V_j having odd number of vertices, the edges between V_i and V_j can be decomposed into only single edges and so the edge set $E(G)$ is the only $\mathcal{ED}$ of G in this case. Thus $\pi_{pi}(G) = m$ when there are at least two parts of odd order. On the other hand, when at most one part in $(V_1, V_2, \ldots, V_k)$ is of odd order, the edges between every pair of parts V_i and V_j can be decomposed into induced paths of length two so that $\pi_{pi}(G) \leqslant \dfrac{m}{2}$. Further, since the length of an induced path in G is at most two we need at least $\dfrac{m}{2}$ induced paths to decompose G and hence $\pi_{pi}(G) \geqslant \dfrac{m}{2}$. Thus $\pi_{pi}(G) = \dfrac{m}{2}$ when at most one part is of odd order. □

Corollary 2.10 *For the complete bipartite graph $K_{r,s}$ we have*

$$\pi_{pi}(K_{r,s}) = \begin{cases} rs & \text{if } rs \text{ is odd} \\ \dfrac{rs}{2} & \text{if } rs \text{ is even} \end{cases}$$

Proof When at most one of the values of r and s is odd, rs is even and it is odd when both r and s are odd. Therefore the result follows by Theorem 2.9. □

For integers s and k with $s \geq 3$ and $0 < k < \dfrac{s}{2}$, the *generalized Petersen graph* $P(s,k)$ is the simple graph with vertices $\{u_i, v_i \,:\, 1 \leq i \leq s\}$ and edges $\{u_iu_{i+1}, u_iv_i, v_iv_{i+k}\}$ where the addition is modulo s.

Theorem 2.11 *For the generalized Petersen graph $P(s,k)$, the value of $\pi_{pi}(P(s,k))$ is $\dfrac{n}{2}$.*

Proof Obviously the generalized Petersen Graph $P(s,k)$ is a three regular graph of order $2s$ and size $3s$. Therefore by Corollary 2.7 we have $\pi_{pi}(P(s,k) \geqslant s$. Let $P_i = (u_i, u_{i+1}, v_{i+1}, v_{i+1+k})$; $1 \leq i \leq s$, where addition is modulo s. Then P_i is an induced path of length 3 and $\psi = \{P_1, P_2, P_3, \ldots, P_s\}$ is an $\mathcal{ED}$ for $P(s,k)$ so that $\pi_{pi}(P(s,k)) \leqslant |\psi| = s = \dfrac{m}{3} = \dfrac{n}{2}$ and hence we obtain the desired result. □

Theorem 2.12 *For the fan $F_n = P_{n-1} + K_1$ with $n > 2$,*

$$\pi_{pi}(F_n) = \begin{cases} n & \text{when } n \text{ is odd} \\ n+1 & \text{when } n \text{ is even} \end{cases}$$

Proof Let $V(F_n) = \{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$ where the vertex v_n correspond to K_1 and $E(F_n) = \{v_iv_{i+1} : 1 \leqslant i \leqslant n-2\} \cup \{v_iv_n : i = 1, 2, 3, \ldots, n-1\}$. Since the size of F_n is always odd, any $\mathcal{ED}$ of F_n is an $O\mathcal{ED}$. Now, let

$$Q_i = (v_i, v_n), \quad \textit{for all } i = 1, 2, 3, \ldots, n-1,$$
$$Q_n = (v_1, v_2, v_3, \ldots, v_{n-2}) \text{ and}$$
$$Q_{n+1} = (v_{n-2}, v_{n-1}).$$

Suppose n is odd. Then $\psi_1 = \{Q_1, Q_2, Q_3, \ldots, Q_{n-1}, P_{n-1}\}$ is an $O\mathcal{ED}$ of F_n so that $\pi_{pi}(F_n) \leqslant |\psi_1| = n$. Moreover, the induced paths containing v_n are of length at most two and hence any $O\mathcal{ED}$ of F_n includes all the $n-1$ edges incident at v_n. In addition, we need at least one more path to cover the remaining edges which lie on the path P_{n-1} and hence $\pi_{pi}(F_n) \geqslant n-1+1 = n$. Thus we get $\pi_{pi}(F_n) = n$. If n is even, then $\psi_2 = \{Q_1, Q_2, Q_3, \ldots, Q_{n-1}, Q_n, Q_{n+1}\}$ is an $O\mathcal{ED}$ of F_n. So that $\pi_{pi}(F_n) \leqslant |\psi_2| = n+1$. A similar argument shows that $\pi_{pi}(F_n) = n+1$. □

Theorem 2.13 *For the double fan $G = P_n + (\overline{K_2})$ with $n \geqslant 2$,*

$$\pi_{pi}(G) = \begin{cases} n+1 & \textit{if } n \textit{ is odd} \\ 2n+1 & \textit{if } n \textit{ is even} \end{cases}$$

Proof Let $V(G) = \{u_1, u_2, v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$ and $E(G) = \{v_iv_{i+1} : 1 \leqslant i \leqslant n-1\} \cup \{u_1v_i : i = 1, 2, 3, \ldots, n\} \cup \{u_2v_i : i = 1, 2, 3, \ldots, n\}$.

Assume that n is odd. Let $Q_i = (u_1, v_i, u_2)$ for all $i = 1, 2, 3, \ldots, n$ and $\psi_1 = \{Q_1, Q_2, Q_3, \ldots, Q_n, P_n\}$. Then ψ_1 is an $E\mathcal{ED}$ of G with cardinality $n+1$ so that $\pi_{pi}(G) \leqslant n+1$. Further, the induced paths of G other than P_n are of length at most two and hence at least n paths of length two are necessary to cover the edges $u_1v_i (i = 1, 2, 3, \ldots, n)$ and $u_2v_i (i = 1, 2, 3, \ldots, n)$. Therefore if ψ is any $\mathcal{ED}$ of G then $|\psi| \geqslant n+1$ and hence we get $\pi_{pi}(G) = n+1$ when n is odd. Now, suppose that n is even. Then size of G is odd which implies that any $\mathcal{ED}$ of G will be an $O\mathcal{ED}$. Since P_n is an odd path in G, P_n along with the remaining edges of G forms an $O\mathcal{ED}$ with cardinality $2n+1$ so that $\pi_{pi}(G) \leqslant 1+2n$. Moreover, P_n is the only odd path in G with length greater than one and hence if ψ is an $O\mathcal{ED}$, it will contain all the edges lying outside P_n. So $|\psi| \geqslant 2n+1$. Thus $\pi_{pi}(G) = 2n+1$ when n is even. □

§3. Bounds for π_{pi}

In this section we obtain some bounds for π_{pi} of a graph in terms of some known graph theoretic parameters. Also we discuss the relation of π_{pi} with some existing decomposition parameters. First we present bounds of π_{pi} in terms of the diameter and the girth of a graph.

Theorem 3.1 *For any graph G with diameter d,*

$$\pi_{pi}(G) \leqslant \begin{cases} m-d+1 & \textit{if } d \textit{ is odd} \\ m-d+2 & \textit{if } d \textit{ is even} \end{cases}$$

Proof Let P be a diameter path(a path whose length is the diameter of the graph) in G. Then P is an induced path of length d. If d is odd, then, the path P together with the remaining edges of G form an $\mathcal{OED}$ of G so that $\pi_{pi}(G) \leqslant m - d + 1$. When d is even, the path $P^{'}$ of length $d-1$ obtained by deleting an edge from P is an odd path and hence G will have an $\mathcal{OED}$ ψ consisting of $P^{'}$ and the remaining edges of G with $|\psi| = 1 + m - (d-1) = m - d + 2$ which gives the desired bound. □

Theorem 3.2 *If G is a graph with girth g, then*

$$\pi_{pi}(G) \leq \begin{cases} m - g + 3 & \textit{if } g \textit{ is odd} \\ m - g + 4 & \textit{if } g \textit{ is even} \end{cases}$$

Proof Let C be the shortest cycle in G of length g. Let P be the path obtained from C by deleting a path of length two. Then P is an induced path. By a similar argument followed in Theorem 3.1 the desired result follows. □

Remark 3.3 The bounds given in Theorem 3.1 and Theorem 3.2 are attained for several classes of graphs. For example, it can be easily verified that the complete graphs and complete multipartite graphs in which at most one partition is consisting of an odd number of vertices are such classes of graphs.

As observed in Remark 3.3, one can list several classes of graphs attaining the bounds given in the above theorems; which means the class of those graphs is relatively larger and so the following problems are worth trying.

Problem 3.4 *Characterize the graphs for which*

(*i*) $\pi_{pi} = m - d + 1$ *when d is odd;*
(*ii*) $\pi_{pi} = m - d + 2$ *when d is even;*
(*iii*) $\pi_{pi} = m - g + 3$ *when g is odd;*
(*iv*) $\pi_{pi} = m - g + 4$ *when g is even.*

Now, it is obvious that the value of π_{pi} of a graph G is ranging from 1 to m where m is the size of G and the lower bound is attained only for paths. On the other hand there are infinitely many graphs attaining the upper bound m. A simple example is a class of complete multipartite graphs as in Theorem 2.9 and the following is another such an infinite family.

Example 3.5 Let G be the graph obtained by pasting two complete graphs at an edge. For example pasting two triangles we get K_4 minus an edge. Now, if $e = (u, v)$ is the edge at which the complete graphs K_r and K_s are pasted, then u and v are adjacent to all the vertices of G. Since the induced paths in G are of length at most two and the edge e does not belong to any induced path of length two the only $\mathcal{ED}$ possible for G is that of the edge set of G and hence we have $\pi_{pi}(G) = m$.

Also it follows from Theorem 3.1 that the diameter necessarily be at most 2 for such graphs. That is, the graphs with $\pi_{pi} = m$ are either complete or of diameter 2. However, the problem

of determining these graphs seems to be a little challenging to settle. As a first step we solve the problem in the case of block graphs.

Theorem 3.6 *If G is a block graph which is not a star of odd order, then $\pi_{pi}(G) = m$ if and only if G contains exactly one cut vertex.*

Proof Suppose G is a block graph which is not a star of odd order with $\pi_{pi}(G) = m$ having more than one cut vertex. Let u and v be two cut vertices of G that are adjacent. Then both u and v belong to the same block, say B_k of G. Let $e_u = w_1u$ and $e_v = vw_2$ be two edges of G belonging to two different blocks other than B_k. Then the path $P = (w_1, u, v, w_2)$ is an induced path of length three so that P together with the remaining edges form an $O\mathcal{ED}$ of G with cardinality less than m contradicting the assumption that $\pi_{pi}(G) = m$. Hence G contains exactly one cut vertex.

Conversely, suppose G contains exactly one cut vertex, say v. If all the blocks of G are of order 2, then $G = K_{1,s}$ where s is odd and so by Theorem 2.9 we have $\pi_{pi}(G) = m$. If not, let B_r be a block of order greater than 2. Let e be an edge of B_r that is not incident at the cut vertex v. Then e does not belong to any induced path of length greater than one. Moreover, the maximum length of any induced path in G is 2. Hence $E(G)$ is the only $\mathcal{ED}$ for G so that $\pi_{pi}(G) = m$ and this completes the proof. □

Corollary 3.7 *If T is a tree, then $\pi_{pi}(T) = m$ if and only if T is a star of even order.*

Proof Notice that $\pi_{pi} = \dfrac{m}{2}$ for a star of odd order. Therefore the result follows from Theorem 3.6. □

In the following we establish some interesting relations between π_{pi} and some existing path decomposition parameters such as the induced acyclic path decomposition number π_{ia} and equiparity path decomposition number π_p. A decomposition of a graph into induced paths is called an *induced path decomposition* and a decomposition into paths of same parity is called *equiparity path decomposition*. The minimum cardinality of such decompositions are denoted by π_{ia} and π_p respectively.

Theorem 3.8 *For any graph G, we have $\pi_{ia}(G) \leqslant \pi_{pi}(G) \leqslant 2\pi_{ia}(G) - 1$. Further if a and b are two positive integers with $a \leqslant b \leqslant 2a - 1$, then there exists a graph G such that $\pi_{ia}(G) = a$ and $\pi_{pi}(G) = b$.*

Proof The first inequality is immediate because every equiparity induced path decomposition will be an induced acyclic path decomposition. Now, let ψ be an induced acyclic path decomposition of G with r paths of even length and s paths of odd length. If either r or s is zero, then $\pi_{ia}(G) = \pi_{pi}(G)$. Assume that both r and s are positive. Now, split each path of even length in ψ into two paths of odd length and obtain a path decomposition ψ' consisting of these paths of odd length along with all the paths of odd length in ψ. Then ψ' will be an $O\mathcal{ED}$ with cardinality $2r + s$ which is obviously at most $2\pi_{ia}(G) - 1$.

Now, let a and b be given integers with $a \leqslant b \leqslant 2a - 1$. We construct a graph G for which $\pi_{ia}(G) = a$ and $\pi_{pi}(G) = b$ as follows. If $a = 1$, then $b = 1$ and so G must be a path.

If $a = 2$ and $b = 2$, let $G = K_{1,4}$ and if $a = 2$ and $b = 3$, let $G = K_{1,3}$. Assume $a \geqslant 3$. Take $b = 2a - 1 - r$ where $0 \leqslant r \leqslant a - 1$. Let G be the graph obtained from the triangle (v_1, v_2, v_3, v_1) by attaching r paths of length 2 along with $2a - 4 - r$ pendant edges at a vertex of the triangle, say v_1. We now prove that $\pi_{ia}(G) = a$ and $\pi_{pi}(G) = b$. Let $x'_1, x'_2, x'_3, \ldots, x'_r$ be the vertices of degree 2 lying outside the triangle and let $x_1, x_2, x_3, \ldots, x_r$ be the pendant vertices adjacent to $x'_1, x'_2, x'_3, \ldots, x'_r$ respectively. Let us denote the remaining pendant vertices of G by $y_1, y_2, y_3, \ldots y_{r-2}, z_1, z_2, z_3, \ldots, z_{2a-2r-2}$ as in Fig.2.

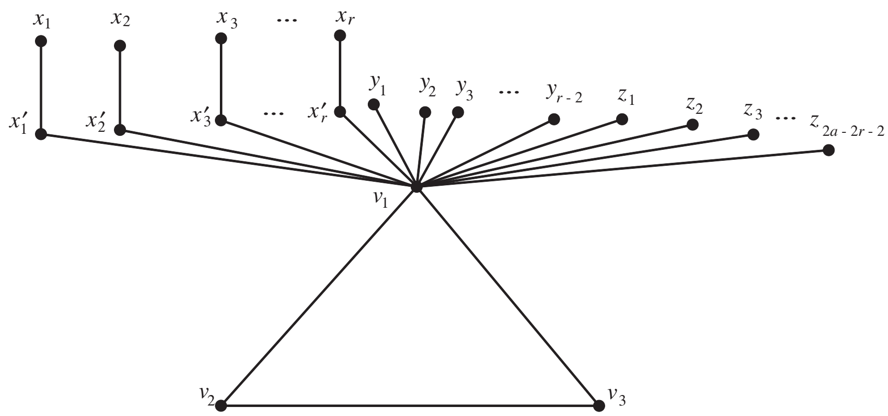

Fig.2

Now let

$$
\begin{aligned}
&P_1 = (x_1, x_1', v_1, v_2), \quad P_2 = (x_2, x_2', v_1, v_3),\\
&P_i = (x_i, x_i', v_1, y_{i-2}) \quad \text{for all } i = 3, 4, 5, \ldots, r\\
&Q_i = (z_i, v_1) \quad \text{for all } i = 1, 2, 3, \ldots, 2a - 2r - 2 \text{ and}\\
&R_i = (z_{2i-1}, v_1, z_{2i}) \quad \text{for all } i = 1, 2, 3, \ldots, a - r - 1.
\end{aligned}
$$

Then $\psi_1 = \{P_1, P_2, \ldots, P_r, R_1, R_2, \ldots R_{a-r-1}, (v_2, v_3)\}$ is an induced acyclic path decomposition of G with $|\psi_1| = a$ so that $\pi_{ia}(G) \leqslant a$. Further, any induced acyclic path decomposition must contain at least $\dfrac{(2a-2)}{2} = a - 1$ induced paths in order to cover all the $2a - 2$ edges of G incident at the vertex v_1 and also none of these induced paths paths cover the edge v_2v_3 as these paths are induced so that $\pi_{ia}(G) \geqslant a$ and thus $\pi_{ia}(G) = a$.

Next we observe that $\psi_2 = \{P_1, P_2, \ldots, P_r, Q_1, Q_2, \ldots Q_{2a-2r-2}, (v_2, v_3)\}$ is an $\mathcal{OED}$ of G with $|\psi_2| = 2a - 1 - r = b$ so that $\pi_{pi}(G) \leqslant b$. Further let ψ be any $\mathcal{ED}$. Since the edge v_2v_3 cannot be a part of any induced path of length greater than one, it itself must be a member of ψ so that ψ is an $\mathcal{OED}$. Hence among the $2a - 2$ edges incident at v_1, only the edges $x_i'v_1, (i = 1, 2, 3, \ldots, r)$ can be a part of an induced path of of length greater than one and each of the remaining $2a - 2 - r$ edges must be a member of ψ so that $|\psi_2| \geqslant 2a - 2 - r + 1 = b$. Thus $\pi_{pi}(G) = b$ and this completes the proof of the theorem. □

Remark 3.9 Since an equiparity induced path decomposition is an equiparity path decomposition and an equiparity path decomposition is a path decomposition, it follows that

$\pi_a(G) \leqslant \pi_p(G) \leqslant \pi_{pi}(G)$ for any graph G. Further these inequalities can be strict. That is, all the three parameters can be either distinct or all are equal. For example, these parameters coincide in the case of paths, cycles of even length and Petersen graph and if $H = G - v_4v_5$, where G is the graph given in Figure 1, then $\pi_a(H) = 2$, $\pi_p(H) = 3$ and $\pi_{pi}(H) = 5$. The the following interpolation problem naturally arises.

Problem 3.10 *If a, b and c are positive integers with $a \leqslant b \leqslant c$ does there exist a graph G such that $\pi_a(G) = a, \pi_p(G) = b$ and $\pi_{pi}(G) = c$?*

§4. Conclusion and Scope

The theory of decomposition is one of the fastest growing areas of research in graph theory. We have come across varieties of decompositions in the literature and most of them are defined by demanding the members of the decomposition to posses some interesting properties. We have introduced the concept of the equiparity induced path decomposition wherein the concepts of equiparity and induceness have been combined. This study is just a first step in this direction. However, there is wide scope for further research on this parameter and here we list some of them.

(1) *Determine the value of π_{pi} for more classes of graphs like trees, unicyclic graphs and bicyclic graphs.*

(2) *Characterize the graphs for which $\pi_{pi} = \frac{m}{2}$, m, π_{ia}, $2\pi_{ia} - 1$ or $\pi_p = \pi_a$.*

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Dual Spacelike Elastic Biharmonic Curves with Timelike Principal Normal According to Dual Bishop Frames

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Abstract: In this paper, we study dual spacelike elastic biharmonic curves with spacelike binormal in dual Lorentzian space $\mathbb{D}_1^3$. We use Noether's Theorem in our main theorem. Finally we obtain Killing vector field acoording to dual spacelike elastic biharmonic curves with spacelike binormal in dual Lorentzian space $\mathbb{D}_1^3$.

Key Words: Dual space curve, dual bishop frame, biharmonic curve.

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§1. Introduction

The Mathematical Theory of Elasticity is occupied with an attempt to reduce to calculation the state of strain, or relative displacement, within a solid body which is subject to the action of an equilibrating system of forces, or is in a state of slight internal relative motion, and with endeavours to obtain results which shall be practically important in applications to architecture, engineering, and all other useful arts in which the material of construction is solid. Its history should embrace that of the progress of our experimental knowledge of the behaviour of strained bodies, so far as it has been embodied in the mathematical theory, of the development of our conceptions in regard to the physical principles necessary to form a foundation for theory, of the growth of that branch of mathematical analysis in which the process of the calculations consists, and of the gradual acquisition of practical rules by the interpretation of analytical results.

Elastic structures con ned to a certain volume or area appear in many situations. For example inner membranes in biological cells separate an inner region from the rest of the cell and consist of an elastic bilayer. The inner structures are con ned by the outer cell membrane. Since the inner membrane contributes to the biological function it is advantageous to include a large membrane area in the cell.

In this paper, we study dual spacelike elastic biharmonic curves with spacelike binormal in dual Lorentzian space $\mathbb{D}_1^3$. We use Noether's Theorem in our main theorem. Finally we obtain Killing vector field acoording to dual spacelike elastic biharmonic curves with spacelike binormal in dual Lorentzian space $\mathbb{D}_1^3$.

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§2. Preliminaries

If φ and φ^* are real numbers and $\varepsilon^2 = 0$ the combination $\hat{\varphi} = \varphi + \varphi^*$ is called a *dual number*. The symbol ε designates the dual unit with the property $\varepsilon^2 = 0$. In analogy with the complex numbers W.K. Clifford defined the dual numbers and showed that they form an algebra, not a field. Later, E.Study introduced the dual angle subtended by two nonparallel lines $\mathbb{E}^3$, and defined it as $\hat{\varphi} = \varphi + \varphi^*$ in which φ and φ^* are, respectively, the projected angle and the shortest distance between the two lines.

In the Euclidean 3-Space $\mathbb{E}^3$, lines combined with one of their two directions can be represented by unit dual vectors over the the ring of dual numbers. The important properties of real vector analysis are valid for the dual vectors. The oriented lines $\mathbb{E}^3$ are in one to one correspondence with the points of the dual unit sphere $\mathbb{D}^3$.

A dual point on $\mathbb{D}^3$ corresponds to a line in $\mathbb{E}^3$, two different points of $\mathbb{D}^3$ represents two skew lines in $\mathbb{E}^3$. A differentiable curve on $\mathbb{D}^3$ represents a ruled surface $\mathbb{E}^3$. The set

$$\mathbb{D}^3 = \left\{\hat{\varphi} : \hat{\varphi} = \varphi + \varepsilon\varphi^*,\ \varphi, \varphi^* \in \mathbb{E}^3\right\}$$

is a module over the ring $\mathbb{D}$.

The elements of $\mathbb{D}^3$ are called *dual vectors*. Thus a dual vector $\hat{\varphi}$ can be written

$$\hat{\mathbf{\Omega}} = \mathbf{\Omega} + \varepsilon\mathbf{\Omega}^*,$$

where φ and φ^* are real vectors in $\mathbb{R}^3$.

The *Lorentzian inner product* of dual vectors $\hat{\varphi}$ and $\hat{\psi}$ in $\mathbb{D}^3$ is defined by

$$\left\langle \hat{\mathbf{\Omega}}, \hat{\psi} \right\rangle = \langle \mathbf{\Omega}, \psi \rangle + \varepsilon \left(\langle \mathbf{\Omega}, \psi^* \rangle + \langle \mathbf{\Omega}^*, \psi \rangle \right),$$

with the Lorentzian inner product φ and ψ

$$\langle \mathbf{\Omega}, \psi \rangle = -\Omega_1\psi_1 + \Omega_2\psi_2 + \Omega_3\psi_3,$$

where $\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ and $\psi = (\psi_1, \psi_2, \psi_3)$. Therefore, $\mathbb{D}^3$ with the Lorentzian inner product $\left\langle \hat{\mathbf{\Omega}}, \hat{\psi} \right\rangle$ is called *3-dimensional dual Lorentzian space* and denoted by of $\mathbb{D}^3_1$. For $\hat{\mathbf{\Omega}} \neq 0$, the *norm* $\left\|\hat{\mathbf{\Omega}}\right\|$ of $\hat{\mathbf{\Omega}}$ is defined by

$$\left\|\hat{\mathbf{\Omega}}\right\| = \sqrt{\left\langle \hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}} \right\rangle}.$$

A dual vector $\hat{\mathbf{\Omega}} = \varphi + \varepsilon\varphi^*$ is called *dual spacelike vector* if $\left\langle \hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}} \right\rangle > 0$ or $\hat{\mathbf{\Omega}} = \mathbf{0}$, *dual timelike vector* if $\left\langle \hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}} \right\rangle < 0$ and *dual null (lightlike) vector* if $\left\langle \hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}} \right\rangle = 0$ for $\hat{\mathbf{\Omega}} \neq 0$.

Therefore, an arbitrary dual curve, which is a differentiable mapping onto $\mathbb{D}^3_1$, can locally be dual space-like, dual time-like or dual null, if its velocity vector is respectively, dual spacelike, dual timelike or dual null.

§3. Spacelike Dual Biharmonic Curves with Spacelike Principal Normal in the Dual Lorentzian Space $\mathbb{D}_1^3$

Let $\hat{\gamma} = \gamma + \varepsilon\gamma^* : I \subset R \to \mathbb{D}_1^3$ be a C^4 dual spacelike curve with spacelike principal normal by the arc length parameter s. Then the unit tangent vector $\hat{\gamma}' = \hat{\mathbf{t}}$ is defined, and the principal normal is $\hat{\mathbf{n}} = \frac{1}{\hat{\kappa}}\nabla_{\hat{\mathbf{t}}}\hat{\mathbf{t}}$, where $\hat{\kappa}$ is never a pure-dual. The function $\hat{\kappa} = \left\|\nabla_{\hat{\mathbf{t}}}\hat{\mathbf{t}}\right\| = \kappa + \varepsilon\kappa^*$ is called the dual curvature of the dual curve $\hat{\gamma}$. Then the binormal of $\hat{\gamma}$ is given by the dual vector $\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$. Hence, the triple $\left\{\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}}\right\}$ is called the Frenet frame fields and the Frenet formulas may be expressed

$$\begin{aligned} \nabla_{\hat{\mathbf{t}}}\hat{\mathbf{t}} &= \hat{\kappa}\hat{\mathbf{n}}, \\ \nabla_{\hat{\mathbf{t}}}\hat{\mathbf{n}} &= \hat{\kappa}\hat{\mathbf{t}} + \hat{\tau}\hat{\mathbf{b}}, \\ \nabla_{\hat{\mathbf{t}}}\hat{\mathbf{b}} &= \hat{\tau}\hat{\mathbf{n}}, \end{aligned} \tag{3.1}$$

where $\hat{\tau} = \tau + \varepsilon\tau^*$ is the dual torsion of the timelike dual curve $\hat{\gamma}$. Here, we suppose that the dual torsion $\hat{\tau}$ is never pure-dual. In addition,

$$\begin{aligned} g\left(\hat{\mathbf{t}}, \hat{\mathbf{t}}\right) &= 1,\ g\left(\hat{\mathbf{n}}, \hat{\mathbf{n}}\right) = -1,\ g\left(\hat{\mathbf{b}}, \hat{\mathbf{b}}\right) = 1, \\ g\left(\hat{\mathbf{t}}, \hat{\mathbf{n}}\right) &= g\left(\hat{\mathbf{t}}, \hat{\mathbf{b}}\right) = g\left(\hat{\mathbf{n}}, \hat{\mathbf{b}}\right) = 0. \end{aligned} \tag{3.2}$$

In the rest of the paper, we suppose everywhere $\hat{\kappa} \neq 0$ and $\hat{\tau} \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} &\nabla_{\hat{\mathbf{t}}}\hat{\mathbf{t}} = \hat{k}_1\hat{\mathbf{m}}_1 - \hat{k}_2\hat{\mathbf{m}}_2, \\ &\nabla_{\hat{\mathbf{t}}}\hat{\mathbf{m}}_1 = \hat{k}_1\hat{\mathbf{t}}, \\ &\nabla_{\hat{\mathbf{t}}}\hat{\mathbf{m}}_2 = \hat{k}_2\hat{\mathbf{t}}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} &g\left(\hat{\mathbf{t}}, \hat{\mathbf{t}}\right) = 1,\ g\left(\hat{\mathbf{m}}_1, \hat{\mathbf{m}}_1\right) = -1,\ g\left(\hat{\mathbf{m}}_2, \hat{\mathbf{m}}_2\right) = 1, \\ &g\left(\hat{\mathbf{t}}, \hat{\mathbf{m}}_1\right) = g\left(\hat{\mathbf{t}}, \hat{\mathbf{m}}_2\right) = g\left(\hat{\mathbf{m}}_1, \hat{\mathbf{m}}_2\right) = 0. \end{aligned} \tag{3.4}$$

Here, we shall call the set $\{\hat{\mathbf{t}}, \hat{\mathbf{m}}_1, \hat{\mathbf{m}}_1\}$ as Bishop trihedra, $\hat{k}_1$ and $\hat{k}_2$ as Bishop curvatures. Here $\tau(s) = \hat{\theta}'(s)$ and $\hat{\kappa}(s) = \sqrt{\left|\hat{k}_2^2 - \hat{k}_1^2\right|}$. Thus, Bishop curvatures are defined by

$$\begin{aligned} \hat{k}_1 &= \hat{\kappa}(s)\sinh\hat{\theta}(s), \\ \hat{k}_2 &= \hat{\kappa}(s)\cosh\hat{\theta}(s). \end{aligned} \tag{3.5}$$

Theorem 3.1 *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a non-geodesic spacelike dual curve with spacelike binormal parametrized by arc length. $\hat{\gamma}$ is a non-geodesic spacelike dual biharmonic curve if and only if*

$$\begin{aligned} &\hat{k}_1^2 - \hat{k}_2^2 = \hat{\Omega}, \\ &\hat{k}_1'' + \hat{k}_1^3 - \hat{k}_2^2\hat{k}_1 = 0, \\ &-\hat{k}_2'' + \hat{k}_2^3 - \hat{k}_1^2\hat{k}_2 = 0, \end{aligned} \tag{3.6}$$

where $\hat{\Omega}$ is dual constant of integration, [5].

Lemma 3.2 *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a non-geodesic spacelike dual curve with spacelike binormal parametrized by arc length. $\hat{\gamma}$ is a non-geodesic spacelike dual biharmonic curve if and only if*

$$\begin{aligned} \hat{k}_1^2 - \hat{k}_2^2 &= \hat{\Omega}, \\ \hat{k}_1'' + \hat{k}_1\hat{\Omega} &= 0, \\ \hat{k}_2'' + \hat{k}_2\hat{\Omega} &= 0, \end{aligned} \tag{3.7}$$

where $\hat{\Omega} = \Omega + \varepsilon\Omega^*$ is constant of integration, [5].

Corollary 3.3 *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a non-geodesic spacelike dual curve with spacelike binormal parametrized by arc length. $\hat{\gamma}$ is a non-geodesic spacelike dual biharmonic curve if and only if*

$$k_1^2 - k_2^2 = -\Omega, \tag{3.8}$$

$$k_1k_1^* - k_2k_2^* = -\Omega^*. \tag{3.9}$$

§4. Dual Spacelike Elastic Biharmonic Curves with Timelike Normal in the Dual Lorentzian Space $\mathbb{D}_1^3$

Consider regular curve (curves with nonvanishing velocity vector) in dual Lorentzian space $\mathbb{D}_1^3$ defined on a fixed interval $I = [a_1, a_2]$:

$$\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3.$$

We will assume (for technical reasons) that the curvature $\hat{\kappa}$ of $\hat{\gamma}$ is nonvanishing. The elastica minimizes the bending energy

$$\Pi(\hat{\gamma}) = \int_{\hat{\gamma}} \hat{\kappa}(s)^2 \, ds$$

with fixed length and boundary conditions. Accordingly, let α_1 and α_2 be points in $\mathbb{D}_1^3$ and α_1', α_2' nonzero vectors. We will consider the space of smooth curves

$$\Xi = \{\hat{\gamma} : \hat{\gamma}(a_i) = \hat{\alpha}_i,\ \hat{\gamma}'(a_i) = \hat{\alpha}_i'\},$$

and the subspace of unit-speed curves

$$\Xi_u = \{\hat{\gamma} \in \Omega : \|\hat{\gamma}'\| = 1\}.$$

Later on we need to pay more attention to the precise level of differentiability of curves, but we will ignore that for now.

$\Pi^\lambda : \Omega \longrightarrow \mathbb{D}$ is defined by

$$\Pi^\lambda(\hat{\gamma}) = \frac{1}{2}\int_{\hat{\gamma}} \left[\|\hat{\gamma}''\| + \hat{\Lambda}(t)\left(\|\hat{\gamma}'\| - 1\right)\right] dt,$$

where $\hat{\Lambda}(t) = \Lambda(t) + \varepsilon\Lambda^*(t)$ is a pointwise dual multiplier, constraining speed.

Theorem 4.1 *(Noether's Theorem) If $\hat{\gamma}$ is a solution curve and W is an infinitesimal symmetry, then*

$$\hat{\gamma}''.W' + \left(\hat{\Lambda}\hat{\gamma}' - \hat{\gamma}'''\right).W$$

is constant. In particular, for a translational symmetry, W is constant; so

$$\left(\hat{\Lambda}\hat{\gamma}' - \hat{\gamma}'''\right).W = \text{constant}.$$

Letting W range over all translations, we get

$$\hat{\Lambda}\hat{\gamma}' - \hat{\gamma}''' = \hat{J}, \tag{4.1}$$

for $\hat{J}$ some constant field and

$$\hat{J} = J + \varepsilon J^*$$

Theorem 4.2 *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a dual spacelike elastic biharmonic curves with spacelike binormal according to Bishop frame. Then,*

$$\Lambda(s) = 0 \text{ and } \Lambda^*(s) = 0. \tag{4.2}$$

Proof Now it is helpful to assume dual biharmonic curve $\hat{\gamma}$ is parametrized by arclength s. If we use dual Bishop frame (3.3), yields

$$\begin{aligned} \hat{\gamma}' &= \hat{\mathbf{t}} \\ \hat{\gamma}'' &= \hat{k}_1\hat{\mathbf{m}}_1 - \hat{k}_2\hat{\mathbf{m}}_2, \\ \hat{\gamma}''' &= \left(\hat{k}_1^2 - \hat{k}_2^2\right)\hat{\mathbf{t}} + \hat{k}_1'\hat{\mathbf{m}}_1 - \hat{k}_2'\hat{\mathbf{m}}_2. \end{aligned} \tag{4.3}$$

By means of dual function, $\varepsilon^2 = 0$ reduces to

$$\begin{aligned} \hat{\gamma}' &= \mathbf{t} + \varepsilon\mathbf{t}^*, \\ \hat{\gamma}'' &= k_1\mathbf{m}_1 - k_2\mathbf{m}_2 + \varepsilon(k_1^*\mathbf{m}_1 + k_1\mathbf{m}_1^* - k_2^*\mathbf{m}_2 - k_2\mathbf{m}_2^*), \\ \hat{\gamma}''' &= \left(k_1^2 - k_2^2\right)\mathbf{t} + k_1'\mathbf{m}_1 - k_2'\mathbf{m}_2 + \varepsilon(\left(k_2^2 - k_1^2\right)\mathbf{t}^* \\ &\quad + \left(2k_2k_2^* - 2k_1k_1^*\right)\mathbf{t} + k_1^{*\prime}\mathbf{m}_1 + k_1'\mathbf{m}_1^* - k_2^{*\prime}\mathbf{m}_2 - k_2'\mathbf{m}_2^*. \end{aligned} \tag{4.4}$$

If we calculate the real and dual parts of this equation, we get the following relations

$$\begin{aligned} \gamma' &= \mathbf{t}, \\ \gamma'' &= k_1\mathbf{m}_1 - k_2\mathbf{m}_2, \\ \gamma''' &= \left(k_2^2 - k_1^2\right)\mathbf{t} + k_1'\mathbf{m}_1 - k_2'\mathbf{m}_2, \end{aligned}$$

and

$$\begin{aligned} \gamma^{*\prime} &= \mathbf{t}^*, \\ \gamma^{*\prime\prime} &= k_1^*\mathbf{m}_1 + k_1\mathbf{m}_1^* - k_2^*\mathbf{m}_2 - k_2\mathbf{m}_2^*, \\ \gamma^{*\prime\prime\prime} &= \left(k_2^2 - k_1^2\right)\mathbf{t}^* + \left(2k_2k_2^* - 2k_1k_1^*\right)\mathbf{t} \\ &\quad + k_1^{*\prime}\mathbf{m}_1 + k_1'\mathbf{m}_1^* - k_2^{*\prime}\mathbf{m}_2 - k_2'\mathbf{m}_2^*. \end{aligned}$$

Using (4.1), we get

$$\begin{aligned} J &= \left(k_1^2-k_2^2-\Lambda\right)\mathbf{t}+k_1'\mathbf{m}_1-k_2'\mathbf{m}_2, \\ J^* &= \left(k_1^2-k_2^2-\Lambda\right)\mathbf{t}^*+\left(-2k_2k_2^*+2k_1k_1^*-\Lambda^*\right)\mathbf{t} \\ &\quad +k_1^{*\prime}\mathbf{m}_1+k_1'\mathbf{m}_1^*-k_2^{*\prime}\mathbf{m}_2-k_2'\mathbf{m}_2^*. \end{aligned} \tag{4.5}$$

If we take the derivative of $\hat{J}$ with respect to s, we get

$$\begin{aligned} \hat{J}_s &= \left(-\Lambda_s+k_1'k_1-k_2'k_2\right)\mathbf{t}+\varepsilon[\left(-\Lambda_s+k_1'k_1-k_2'k_2\right)\mathbf{t}^* \\ &\quad +(-\Lambda_s^*-k_2'k_2^*-k_2^{*\prime}k_2+k_1'k_1^*+k_1^{*\prime}k_1)\mathbf{t}] \\ &\quad +[k_1''-k_1\left(k_1^2-k_2^2-\Lambda\right)]\mathbf{m}_1 \\ &\quad +\varepsilon[k_1^{*\prime\prime}-\left(k_1^2-k_2^2-\Lambda\right)k_1^*+\left(-2k_2k_2^*+2k_1k_1^*-\Lambda^*\right)k_1]\mathbf{m}_1 \\ &\quad +\varepsilon[k_1''-k_1\left(k_1^2-k_2^2-\Lambda\right)]\mathbf{m}_1^* \\ &\quad -[k_2''-k_2\left(k_1^2-k_2^2-\Lambda\right)]\mathbf{m}_2 \\ &\quad -\varepsilon[k_2^{*\prime\prime}-\left(k_1^2-k_2^2-\Lambda\right)k_2^*+\left(-2k_2k_2^*+2k_1k_1^*-\Lambda^*\right)k_2]\mathbf{m}_2 \\ &\quad -\varepsilon[k_2''-k_2\left(k_1^2-k_2^2-\Lambda\right)]\mathbf{m}_2^*. \end{aligned} \tag{4.6}$$

Then we calculate the real and dual parts of this equation, we get the following relations

$$\begin{aligned} J_s &= \left(-\Lambda_s+k_1'k_1-k_2'k_2\right)\mathbf{t}+[k_1''-k_1\left(k_1^2-k_2^2-\Lambda\right)]\mathbf{m}_1 \\ &\quad -[k_2''-k_2\left(k_1^2-k_2^2-\Lambda\right)]\mathbf{m}_2, \\ J_s^* &= \left(-\Lambda_s+k_1'k_1-k_2'k_2\right)\mathbf{t}^*+\left(-\Lambda_s^*-k_2'k_2^*-k_2^{*\prime}k_2+k_1'k_1^*+k_1^{*\prime}k_1\right)\mathbf{t} \\ &\quad +[k_1^{*\prime\prime}-\left(k_1^2-k_2^2-\Lambda\right)k_1^*+\left(-2k_2k_2^*+2k_1k_1^*-\Lambda^*\right)k_1]\mathbf{m}_1 \\ &\quad +[k_1''-k_1\left(k_1^2-k_2^2-\Lambda\right)]\mathbf{m}_1^* \\ &\quad -[k_2^{*\prime\prime}-\left(k_1^2-k_2^2-\Lambda\right)k_2^*+\left(-2k_2k_2^*+2k_1k_1^*-\Lambda^*\right)k_2]\mathbf{m}_2 \\ &\quad -[k_2''-k_2\left(k_1^2-k_2^2-\Lambda\right)]\mathbf{m}_2^*. \end{aligned}$$

Thus, by taking into consideration that (3.3) and (3.4), we complete the proof. □

Corollary 4.3 *Let $\hat{\gamma}: I \longrightarrow \mathbb{D}_1^3$ be a dual spacelike elastic biharmonic curves with timelike*

binormal according to Bishop frame. Then,

$$\begin{aligned} J_s &= (k_1'k_1 - k_2'k_2)\,\mathbf{t} + [k_1'' - k_1\left(k_1^2 - k_2^2\right)]\mathbf{m}_1 \\ &\quad -[k_2'' - k_2\left(k_1^2 - k_2^2\right)]\mathbf{m}_2 \end{aligned}$$

$$\begin{aligned} J_s^* &= (k_1'k_1 - k_2'k_2)\,\mathbf{t}^* + (-k_2'k_2^* - k_2^{*\prime}k_2 + k_1'k_1^* + k_1^{*\prime}k_1)\mathbf{t} \\ &\quad +[k_1^{*\prime\prime} - \left(k_1^2 - k_2^2\right)k_1^* + \left(-2k_2k_2^* + 2k_1k_1^*\right)k_1]\mathbf{m}_1 \\ &\quad +[k_1'' - k_1\left(k_1^2 - k_2^2\right)]\mathbf{m}_1^* \\ &\quad -[k_2^{*\prime\prime} - \left(k_1^2 - k_2^2\right)k_2^* + \left(-2k_2k_2^* + 2k_1k_1^*\right)k_2]\mathbf{m}_2 \\ &\quad -[k_2'' - k_2\left(k_1^2 - k_2^2\right)]\mathbf{m}_2^* \end{aligned} \tag{4.7}$$

Proof Using (4.2) and (4.6), we have (4.7). This completes the proof. □

Corollary 4.4 *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a dual spacelike elastic biharmonic curves with timelike binormal according to Bishop frame. Then $\hat{J}$ is a Killing vector field.*

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Random Walk on a Finitely Generated Monoid

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Abstract: We study the stability of the waiting time of arrival at first point of length n on a finitely generated monoid. As an example we show that the asymptotic behavior $\psi(n)$ of the average waiting time of arrival at the first element of length n on a monogenic monoid is $\psi(n) \asymp n\ln(n)$, and that of a finitely generated free monoid of at least two generators is $\psi(n) \asymp n$.

Key Words: Random walk, monoid, group, probability, Markov chain

AMS(2010): 11B39, 33C05

§1. Introduction

The study of random walks on finitely generated groups and locally compact groups has identified an invariant for these groups, which is the asymptotic behavior $\phi(n)$ of probabilities of return to the origin (for details see [1], [5], [11]-[12]). In 1959 Kesten (see [7]) showed that $\phi(n)$ decays like $\exp(-n)$ if and only if the group G is non amenable. Later Varoupolos proved that $\exp(-n)$ is a lower bound of $\phi(n)$, that is $\phi(n) \succeq \exp(-n)$.

For a simple random walk on a discrete subgroup G of a connected Lie group, three and only three behaviors may occur (see [1], [3]-[4], [6], [12]-[13]):

1. the group G is non amenable and $\phi(n) \asymp \exp(-n)$,
2. the group G is virtually nilpotent of volume growth $V(n) \asymp n^d$, in this case $\phi(n) \asymp n^{-d/2}$,
3. the group G is virtually polycyclic of exponential volume growth, in this case $\phi(n) \asymp \exp(-n^{1/3})$.

For a large class of solvable groups (see [2], [5], [8]-[10]) the random walk decays like $\exp(-n^{1/3})$.

In the sequel a monoid is a set with an associative internal composition law and has a neutral element denoted by e. In this paper we are interested in the case of monoids and we try to find an invariant in terms of random walks, which does not involve the concept of symmetry.

Let M be a finitely generated monoid, S be a finite minimal generating subset of M. We define for all x of M the length of x by

$$\ell_S(x) = min\{k \in \mathbb{N}; x \in S^k\}$$

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where $S^k = \{x_1 x_2 \cdots x_k | x_i \in S\}$ if $k \neq 0$ and $S^0 = \{e\}$. For a positive integer n we consider $\Omega = (S^n)^{\mathbb{N}}$.
Let $X_i : \Omega \to G$ the i-th canonical projection.

We denote by $A|B$ the event A such that B is realized. We define a probability P on Ω, such that

$$\forall k \in \mathbb{N} \forall g \in M, P(X_{k+1} = g | X_k = g) = \frac{\ell_S(g)}{\mathrm{card}(S^n)},$$

$$\forall g \in M, \forall s \in S, P(X_{k+1} = gs | X_k = g) = \left(1 - \frac{\ell_S(g)}{\mathrm{card}(S^n)}\right) \frac{1}{\mathrm{card}(S)},$$

and $P(X_{k+1} = l | X_k = g) = 0$ in the other cases.

For two real valued functions f, g defined on a discrete subset of $]0, +\infty[$, we define the relation $f \preceq g$ by

$$\exists \alpha, \beta \in]0, +\infty[, \forall x \in]0, +\infty[; \overline{f}(x) \leq \alpha \overline{g}(\beta x) + \alpha,$$

where $\overline{f}$ and $\overline{g}$ are the linear interpolations of f and g. When $f \preceq g$ and $g \preceq f$, we write $f \asymp g$. The asymptotic behavior of f is the equivalence class of f for the relation $\asymp$.

We define the random variable $U_n = \mathrm{card}\{k; X_k \in S^{n-1}\}$, which is the waiting time of arrival at the first element of length n in M. When $U_n = +\infty$ from a certain rank, we say that the random walk on the monoid M is slow. When it is not slow, we are interested in asymptotic behavior of $\psi(n) = E(U_n | X_0 = e)$ when n tends to infinity, which represents the average waiting time of arrival at the first element of length n, starting from the origin. The case of finite monoid is a model on which the random walk is slow since for such a monoid

$$\forall n > \max\{\ell_S(x); x \in M\}; U_n = +\infty.$$

§2. Stability of Asymptotic Behavior of the Average Waiting Time $\phi(n)$

In this section we show that the asymptotic behavior of $\phi(n)$ is independent of the generating set S, which allow us to construct an invariant of the monoid M

Proposition 1 *If S and S' are two minimum generating sets of M, then $\phi_S(n) \asymp \phi_{S'}(n)$.*

Proof Let X'_k be the k-th canonical projection on $\Omega' = (S'^n)^{\mathbb{N}}$ and $U'_n = \mathrm{card}\{i; X'_i \in S'^{n-1}\}$. There exists a positive integer p such that $S \subset S'^p$. By an induction on i, one gets

$$\{i, X_i \in S^{n-1}\} \cap \{X_0 = e\} \subset \{i; X'_i \in S'^{np-1}\} \cap \{X'_0 = e\}.$$

Hence

$$E(U_n | X_0 = e) \leq E(U'_{np} | X'_0 = e).$$

It follows that $\phi(n) \leq \phi(np)$ where $\phi_S(n) \preceq \phi_{S'}(n)$, and exchanging the roles of S and S' we obtain the result. □

§3. **Infinite Monogenic Monoids**

In this section we prove the following result.

Theorem 3.1 *If M is an infinite monogenic monoid, then the average waiting time on M satisfies $\psi(n) \asymp n\ln(n)$.*

Proof The monoid M is monogenic, then there exists $a \in M$ such that $S = \{a\}$ is a minimal generating subset of M. We can write $U_n = \mathrm{card}\{k; \ell_S(X_k) < n\}$. For $k \in \{0, \ldots, n-1\}$,let $u(n,k) = E(U_n | \ell_S(X_0) = k)$, so $\psi(n) = u(n,0)$. Then for all $k \in \{0, \ldots, n-1\}$,

$$\begin{aligned} u(n,k) &= E(U_n|\ell_S(X_0) = k) \\ &= E(U_n|\ell_S(X_0) = k, \ell_S(X_1) = k)P(\ell_S(X_1) = k|\ell_S(X_1) = k) + \\ &\quad E(U_n|\ell_S(X_0) = k, \ell_S(X_1) = k+1)P(\ell_S(X_1) = k+1|\ell_S(X_0) = k) \\ &= \frac{k}{n}u(n,k) + \frac{n-k}{n}u(n,k+1) + 1. \end{aligned}$$

Hence $u(n,k) = u(n,k+1) + \frac{n}{n-k}$ so, $u(n,0) = \sum_0^{n-1} \frac{n}{n-k} \asymp n\ln(n)$ and it follows that $\psi(n) \asymp n\ln(n)$. □

§4. **Free Monoids**

For a free monoid, we have the following result.

Theorem 4.1 *Let M be a free monoid with generators p, $p > 1$. Then the average waiting time of the visit of the n-th ring is $\psi(n) \asymp n$.*

Proof We consider a minimal generating subset $S = \{x_1, \ldots, x_p\}$ of M. Keeping the notations introduced in the preceding section, we have

$$u(n,k) = 1 + \frac{k}{p^n}u(n,k) + (1 - \frac{k}{p^n})u(n,k+1).$$

Therefore

$$u(n,k) - u(n,k+1) = \frac{1}{1 - \frac{k}{p^n}}.$$

Hence $u(n,0) = \sum_{k=0}^{n-1} \frac{1}{1-\frac{k}{p^n}}$, and we obtain $n \le \psi(n) \le n\frac{p^n}{p^n - n + 1}$, and the result follows. □

§5. **Lower and Upper bounds of $\psi(n)$**

We have the following property about the lower bound $\psi(n)$.

Proposition 2 *For any finitely generated monoid M, $\psi(n) \ge n$.*

Proof We have $\psi(n) = E(U_n|X_0 = e)$. When $X_0 = e$ is realized, then $X_1 = X_0$ or $X_1 \in S$ and by induction $X_0, X_1, \ldots, X_{n-1}$ are realized. Consequently, $\{0, 1, \ldots, n-1\} \subset \{i, \ell_S(X_i) < n\}$. So we obtain the lower bound. □

Proposition 3 *For any finitely generated monoid M, for non slow random walk, $\psi(n) \preceq n\ln(n)$.*

Proof We have

$$u(n,k) \leq \frac{k}{\mathrm{card}(S^n)}u(n,k) + (1 - \frac{k}{\mathrm{card}(S^n)})u(n,k+1) + 1$$

and since the random walk on M is not slow then for any n, we have $S^n \subsetneq S^{n+1}$. So for all positive integer n, $\mathrm{card}(S^n) \geq n$, then

$$u(n,0) \leq \sum_{k=0}^{n-1} \frac{\mathrm{card}S^n}{\mathrm{card}S^n - k} \leq \sum_{k=0}^{n-1} \frac{n}{n-k} \leq n(1+\ln(n)).$$ □

§6. Questions

Several questions arise with respect to the profile $\psi(n)$ following.

1. Is there an asymptotic behavior of $\psi(n)$ between n and $n\ln(n)$?

2. For a monoid M, G is the group obtained by symmetrization of M, what is the relationship between the asymptotic behavior of $\psi(n)$ and the probability of return on G?

3. For a non amenable monoid, have we $\psi(n) \asymp n$?

4. What is the asymptotic behavior of $\psi(n)$ in the case of a monoid with polynomial growth of degree d?

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Fibonacci and Super Fibonacci Graceful Labelings of Some Cycle Related Graphs

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Abstract: We investigate Fibonacci and super Fibonacci graceful labelings for some cycle related graphs. We prove that the path union of k-copies of C_m where $m \equiv 0$ $(mod\ 3)$ is a Fibonacci graceful graph. We also discuss the embedding of cycle in the context of these labelings. This work is a nice combination of graph theory and elementary number theory.

Key Words: Graph labeling, Fibonacci graceful, super Fibonacci graceful graph.

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§1. Introduction and Definitions

We begin with simple, finite, undirected and non-trivial graph $G = (V, E)$, with vertex set V and edge set E. In the present work C_n denote the cycle with n vertices and P_n denote the path of n vertices. In the wheel $W_n = C_n + K_1$ the vertex corresponding to K_1 is called the apex vertex and the vertices corresponding to C_n are called the rim vertices where $n \geqslant 3$. Throughout this paper $|V|$ and $|E|$ are used for cardinality of vertex set and edge set respectively. We assume $F_1 = 1, F_2 = 2$ and for each positive integer n, $F_{n+2} = F_{n+1} + F_n$. For each positive integer n, F_n is called the nth Fibonacci number. For various graph theoretic notations and terminology we follow Gross and Yellen [3] while for number theory we follow Burton [1]. We will give brief summary of definitions and other information which are useful for the present investigations.

Definition 1.1 *If the vertices of the graph are assigned values subject to certain conditions then it is known as graph labeling.*

Vast amount of literature is available in printed as well as in electronic form on different types of graph labeling. More than 1200 research papers have been published so far in last four decades. Most interesting graph labeling problems have following three important ingredients.

- a set of numbers from which vertex labels are chosen;

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- a rule that assigns a value to each edge;
- a condition that these values must satisfy.

Most of the graph labeling techniques trace their origin to graceful labeling introduced by Rosa [5].

Definition 1.2 *Let $G = (V, E)$ be a graph with q edges. A graceful labeling of G is an injective function $f : V \to \{0, 1, 2, \cdots, q\}$ such that the induced edge labeling $f(uv) = |f(u) - f(v)|$ is a bijection from E onto the set $\{1, 2, \cdots, q\}$. If a graph G admits a graceful labeling then G is called graceful graph.*

The problem of characterizing all graceful graphs and the graceful tree conjecture provided the reason for different ways of labeling of graphs. Some variations of graceful labeling are also introduced recently such as edge graceful labeling, Fibonacci graceful labeling, odd graceful labeling. For a detailed survey on graph labeling we refer to Gallian [2]. The present work is aimed to discuss Fibonacci graceful labeling.

Definition 1.3 *A Fibonacci graceful labeling of G is an injective function $f : V \to \{0, 1, 2, \cdots, F_q\}$ such that the induced edge labeling $f(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{F_1, F_2, \cdots, F_q\}$. If a graph G admits a Fibonacci graceful labeling then G is called a Fibonacci graceful graph.*

The notion of a Fibonacci graceful labeling was originated by Kathiresan and Amutha [4]. They have proved that K_n is Fibonacci graceful if and only if $n \leqslant 3$ and path P_n is Fibonacci graceful.

Illustration 1.4 The Fibonacci graceful labeling of $K_{1,6}$ and C_6 are shown in Fig.1.

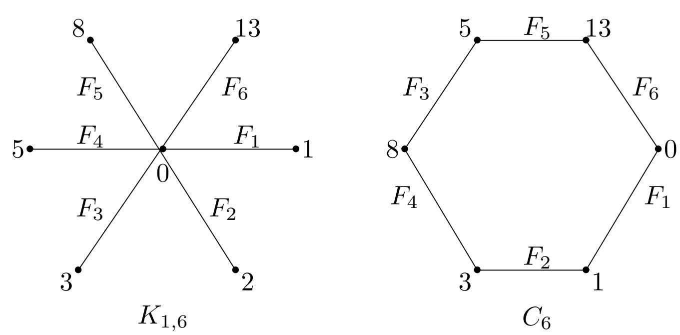

Fig.1

Definition 1.5 *Let $G = (V, E)$ be a graph with q edges. A super Fibonacci graceful labeling of G is an injective function $f : V \to \{0, F_1, F_2, \cdots, F_q\}$ such that the induced edge labeling $f(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{F_1, F_2, \cdots, F_q\}$. If a graph G admits a super Fibonacci graceful labeling then G is called a super Fibonacci graceful graph.*

With reference to Definitions 1.3 and 1.5 we observe that in any (super) Fibonacci graceful

graph there are two vertices having labels 0 and F_q and these vertices are adjacent.

Definition 1.6 *The graph obtained by identifying a vertex of a cycle C_n with a vertex of a cycle C_m is the graph with $|V| = m+n-1$, $|E| = m+n$ and is denoted by $< C_n : C_m >$.*

Definition 1.7 *The graph $G =< C_n : P_k : C_m >$ is the graph obtained by identifying one end vertex of P_k with a vertex of C_n and the other end vertex of P_k with a vertex of C_m.*

Definition 1.8([6]) *Let $G_1, G_2, \cdots, G_k$, $k \geqslant 2$ be k copies of a fixed graph G. Then the graph obtained by joining a vertex of G_i to the corresponding vertex of G_{i+1} by an edge for $i = 1, 2, \cdots, k-1$ is called a path union of $G_1, G_2, \cdots, G_k$.*

Motivated through this definition we define the following.

Definition 1.9 *Let $G_1, G_2, \cdots, G_k$, $k \geqslant 2$ be k graphs of a graph family. Adding an edge between G_i to G_{i+1} for $i = 1, 2, \cdots, k-1$ is called an arbitrary path union of $G_1, G_2, \cdots, G_k$.*

In the next section we investigate some new results on Fibonacci graceful graphs.

§2. Some results on Fibonacci Graceful Graphs

Theorem 2.1 *The graph obtained by joining a vertex of C_{3m} and a vertex of C_{3n} by an edge admits a Fibonacci graceful labeling.*

Proof Let the graph $G =< C_{3m} : P_2 : C_{3n} >$ is obtained by joining a vertex of a cycle C_n with a vertex of a cycle C_m by an edge.

Let the vertices of C_{3m} and C_{3n} in order be $v_0, v_1, v_2, \cdots, v_{3m-1}$ and $u_0, u_1, u_2, \cdots, u_{3n-1}$ respectively. Let u_o and v_0 be joined by an edge e. Then the vertex set of the graph is $V = \{v_0, v_1, v_2, \cdots, v_{3m-1}, u_0, u_1, u_2, \cdots, u_{3n-1}\}$ and the number of edges of G is $|E| = q = 3(m+n)+1$. Define $f : V \longrightarrow \{0, 1, 2, 3, \cdots, F_q\}$ as follows:

$f(v_0) = 0$; for $i = 1, 2, 3, \cdots, 3m-1$,

$$f(v_i) = \begin{cases} F_i & \text{if } i \equiv 1 \ (\mathit{mod}\ 3); \\ F_{i+1} & \text{if } i \equiv 2 \ (\mathit{mod}\ 3); \\ F_{i+2} & \text{if } i \equiv 0 \ (\mathit{mod}\ 3); \end{cases}$$

$f(u_0) = F_q$ and for $i = 1, 2, 3, \cdots, 3n-1$

$$f(u_i) = \begin{cases} F_q + F_{3m+i} & \text{if } i \equiv 1 \ (\mathit{mod}\ 3); \\ F_q + F_{3m+i+1} & \text{if } i \equiv 2 \ (\mathit{mod}\ 3); \\ F_q + F_{3m+i+2} & \text{if } i \equiv 0 \ (\mathit{mod}\ 3). \end{cases}$$

In view of the above defined labeling pattern f admits a Fibonacci graceful labeling for G. That is, G is a Fibonacci graceful graph. □

Illustration 2.2 The Fibonacci graceful labeling of the graph joining a vertex of C_9 and a vertex of C_6 by an edge is as shown in Fig.2.

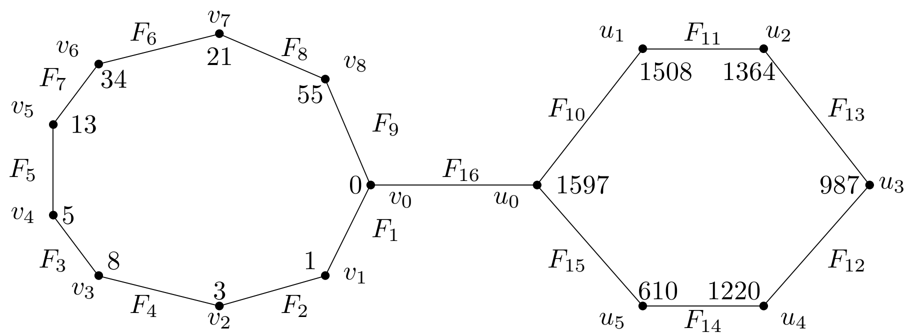

Fig.2

Theorem 2.3 *The graph obtained by joining a vertex of C_{3m} and a vertex of C_{3n} by a path P_3 admits Fibonacci graceful labeling.*

Proof Let the graph $G =< C_{3m} : P_3 : C_{3n} >$ is obtained by joining a vertex of a cycle C_{3m} with a vertex of a cycle C_{3n} by a path P_3.

Let the vertices of C_{3m} and C_{3n} be $v_0, v_1, v_2, \cdots, v_{3m-1}$ and $u_0, u_1, u_2, \cdots, u_{3n-1}$ respectively. Let u_o and v_0 be joined by a path $P_3 = u_0, w_1, v_0$. Here $V = \{v_0, v_1, v_2, \cdots, v_{3m-1}, w_1, u_0, u_1, u_2, \cdots, u_{3n-1}\}$ and the number of edges of G is $|E| = q = 3(m+n)+2$. Define $f : V \longrightarrow \{0, 1, 2, 3, \cdots, F_q\}$ as follows:

$f(v_0) = 0$; for $i = 1, 2, \cdots, 3m-1$,

$$f(v_i) = \begin{cases} F_i & \text{if } i \equiv 1 \ (\mathit{mod}\ 3); \\ F_{i+1} & \text{if } i \equiv 2 \ (\mathit{mod}\ 3); \\ F_{i+2} & \text{if } i \equiv 0 \ (\mathit{mod}\ 3); \end{cases}$$

$f(w_1) = F_q$; $f(u_0) = F_{q-2}$ and for $i = 1, 2, \cdots, 3n-1$,

$$f(u_i) = \begin{cases} F_{q-2} + F_{3m+i} & \text{if } i \equiv 1 \ (\mathit{mod}\ 3); \\ F_{q-2} + F_{3m+i+1} & \text{if } i \equiv 2 \ (\mathit{mod}\ 3); \\ F_{q-2} + F_{3m+i+2} & \text{if } i \equiv 0 \ (\mathit{mod}\ 3). \end{cases}$$

In view of the above defined labeling pattern f admits a Fibonacci graceful labeling of the graph G. That is, G is a Fibonacci graceful graph. $\square$

Illustration 2.4 The Fibonacci graceful labeling of the graph joining a vertex of C_9 and a vertex of C_6 by a path P_3 is as shown in Fig.3.

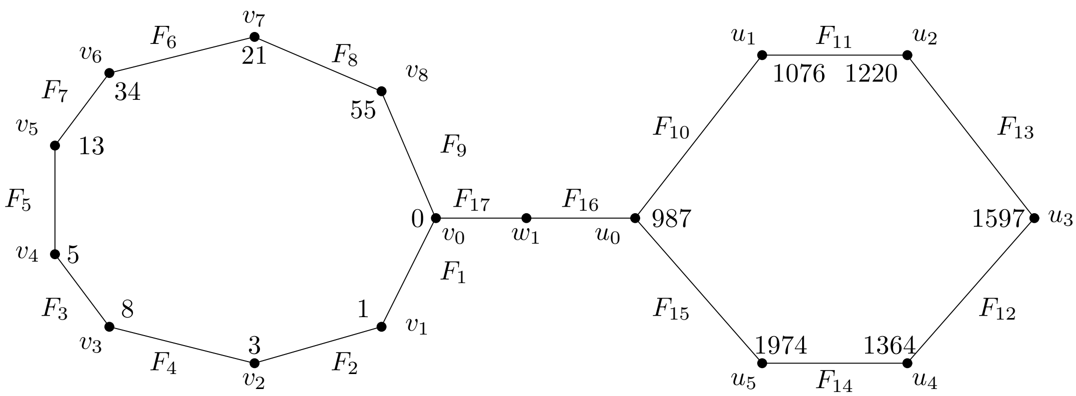

Fig.3

Theorem 2.5 *The graph obtained by joining a vertex of C_{3m} and a vertex of C_{3n} by a path P_k admits Fibonacci graceful labeling.*

Proof Let the graph $G =< C_{3m} : P_k : \ C_{3n} >$ is obtained by joining one vertex of a cycle C_n with one vertex of a cycle C_m by a path of length k.

Let the vertices of C_{3m} and C_{3n} be $v_0, v_1, v_2, \cdots, v_{3m-1}$ and $u_0, u_1, u_2, \cdots, u_{3n-1}$ respectively. Let v_0 and u_0 be joined by a path $P_k = w_0, w_1, w_2, \cdots, w_{k-1}$ on k vertices with $v_0 = w_0$ and $u_0 = w_{k-1}$. The vertex set of G is $V = \{v_0, v_1, \cdots, v_{3m-1}, u_0, u_1, \cdots, u_{3n-1}, w_1, w_2, \cdots, w_{k-2}\}$ and the number of edges of G is $|E| = q = 3(m+n) + k - 1$.

Define $f : V \longrightarrow \{0, 1, 2, 3, \cdots, F_q\}$ as follows:

$f(v_0) = 0$; for $i = 1, 2, 3, \cdots, 3m-1$,

$$f(v_i) = \begin{cases} F_i & \text{if } i \equiv 1 \ (\mathit{mod}\ 3); \\ F_{i+1} & \text{if } i \equiv 2 \ (\mathit{mod}\ 3); \\ F_{i+2} & \text{if } i \equiv 0 \ (\mathit{mod}\ 3); \end{cases}$$

for $i = 1, 2, 3, \cdots, k-1$, $f(w_i) = \sum_{j=1}^{i} (-1)^{j-1} F_{q-(j-1)}$ and for $i = 1, 2, \cdots, 3n-1$,

$$f(u_i) = \begin{cases} f(w_k) + F_{3m+i} & \text{if } i \equiv 1 \ (\mathit{mod}\ 3); \\ f(w_k) + F_{3m+i+1} & \text{if } i \equiv 2 \ (\mathit{mod}\ 3); \\ f(w_k) + F_{3m+i+2} & \text{if } i \equiv 0 \ (\mathit{mod}\ 3). \end{cases}$$

In view of the above defined labeling pattern f admits a Fibonacci graceful labeling of the graph G. That is, G is a Fibonacci graceful graph. □

Illustration 2.6 A Fibonacci graceful labeling of the graph obtained by joining a vertex of C_9 and a vertex of C_6 by a path P_6 is shown in the following Fig.4.

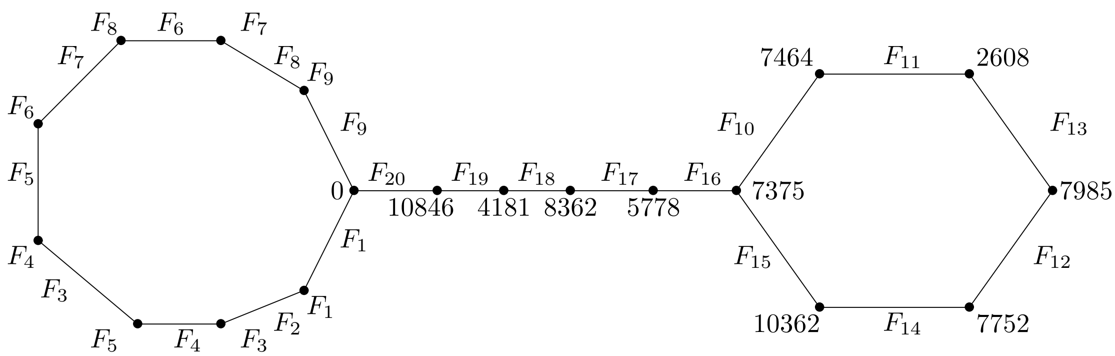

Fig.4

Theorem 2.7 *An arbitrary path union of $k-$copies of cycles C_{3m} is a Fibonacci graceful graph.*

Proof Let the graph G be obtained by attaching cycles $C^i_{3n_i}$ of length $3n_i$ at each of the vertices v_i of a path $P = v_0v_1v_2 \cdots v_{k-1}$ on k vertices. So the number of edges $|E| = q = 3(n_0 + n_1 + \cdots + n_{k-1}) + k - 1$. Let the vertices of each of the cycles $C^i_{3n_i}$ be $u_{i,0}, u_{i,1}, \cdots, u_{i,3n_i-1}$ for each $i = 0, 1, 2, \cdots, k-1$. Let the vertices $u_{0,0}, u_{1,0}, \cdots, u_{k-1,0}$ forms a path $P = u_{0,0}\ u_{1,0}\ \cdots\ u_{k-1,0}$. Define $f : V \longrightarrow \{0, 1, 2, 3, \cdots, F_q\}$ as follows:

$f(u_{0,0}) = 0$; for $i = 1, 2, \cdots, k-1$, $f(u_{i,0}) = \sum_{j=1}^{i}(-1)^{j-1}F_{q-(j-1)}$; for $i = 1, 2, \cdots, n_0 - 1$;

$$f(u_{0,i}) = \begin{cases} F_i & \text{if} \quad i \equiv 1 \quad (\ mod\ 3); \\ F_{i+1} & \text{if} \quad i \equiv 2 \quad (\ mod\ 3); \\ F_{i+2} & \text{if} \quad i \equiv 0 \quad (\ mod\ 3); \end{cases}$$

for $j = 1, 2, \cdots, k-1$ and $i = 1, 2, \cdots, 3n_j - 1$,

$$f(u_{j,i}) = \begin{cases} f(u_{j,0}) + F_{3mj+i} & \text{if} \quad i \equiv 1 \quad (\ mod\ 3); \\ f(u_{j,0}) + F_{3mj+i+1} & \text{if} \quad i \equiv 2 \quad (\ mod\ 3); \\ f(u_{j,0}) + F_{3mj+i+2} & \text{if} \quad i \equiv 0 \quad (\ mod\ 3). \end{cases}$$

In view of the above defined labeling pattern f admits a Fibonacci graceful labeling for graph G. That is, G is a Fibonacci graceful graph. $\square$

Illustration 2.8 In the following Fig.5 the path union of three cycles C_3, C_6 and C_9 with its Fibonacci graceful labeling is shown.

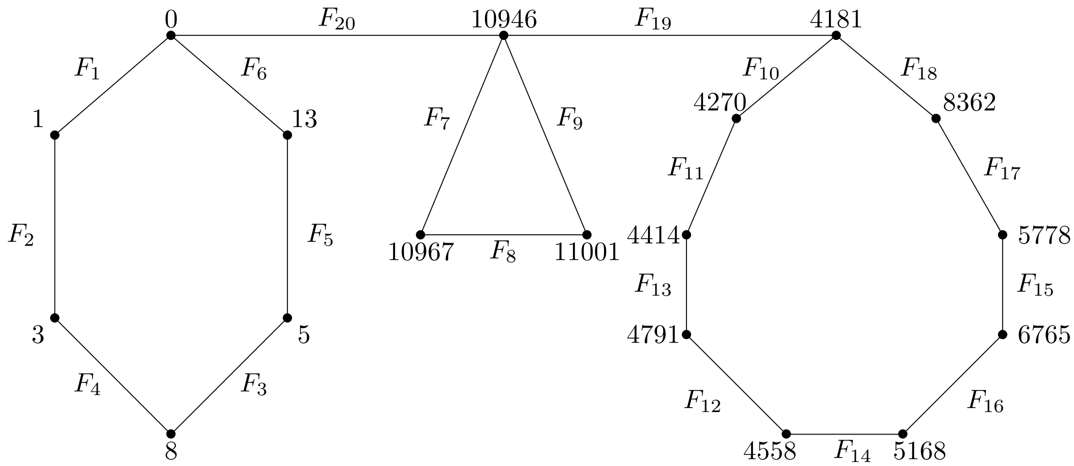

Fig.5

§3. Some Results on Super Fibonacci Graceful Graphs

Theorem 3.1 *One point union of two cycles C_{3m} and C_{3n} is a super Fibonacci graceful graph.*

Proof Let the vertices of C_{3m} and C_{3n} be $v_0, v_1, v_2, \cdots, v_{3m-1}$ and $u_0, u_1, u_2, \cdots, u_{3n-1}$ respectively. One point union of C_{3m} and C_{3n} is obtained by identifying u_0 and v_0. Then the vertex set of the resulting graph G is $V = \{v_0, v_1, v_2, \cdots, v_{3m-1}, u_1, u_2, \cdots, u_{3n-1}\}$ and the number of edges is $|E| = q = 3(m+n)$. Define $f : V \longrightarrow \{0, F_1, F_2, \cdots, F_q\}$ as follows:

$f(v_0) = 0$; for $i = 1, 2, 3, \cdots, 3m-1$

$$f(v_i) = \begin{cases} F_i & \text{if } i \equiv 1 \ (\mathit{mod}\ 3); \\ F_{i+1} & \text{if } i \equiv 2 \ (\mathit{mod}\ 3); \\ F_{i+2} & \text{if } i \equiv 0 \ (\mathit{mod}\ 3); \end{cases}$$

and for $i = 1, 2, 3, \cdots, 3n-1$,

$$f(u_i) = \begin{cases} F_{3m+i} & \text{if } i \equiv 1 \ (\mathit{mod}\ 3); \\ F_{3m+i+1} & \text{if } i \equiv 2 \ (\mathit{mod}\ 3); \\ F_{3m+i+2} & \text{if } i \equiv 0 \ (\mathit{mod}\ 3). \end{cases}$$

In view of the above defined labeling pattern f admits a super Fibonacci graceful labeling of the graph G. That is, G is a super Fibonacci graceful graph. □

Illustration 3.2 The super Fibonacci graceful labeling of $< C_9 : C_6 >$ is as shown in Fig.6.

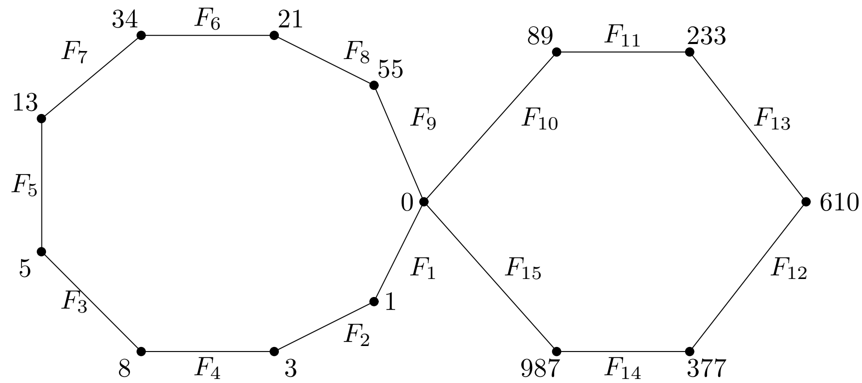

Fig.6

Theorem 3.3 *Every cycle C_n with $n \equiv 0 \ (mod\ 3)$ or $n \equiv 1 \ (mod\ 3)$ is an induced subgraph of a super Fibonacci graceful graph while every cycle C_n with $n \equiv 2 \ (mod\ 3)$ can be embedded as a subgraph of a Fibonacci graceful graph.*

Proof Let the cycle C_n has the n vertices $v_0, v_1, \cdots, v_{n-1}$ in order. For the positive integer $n \geqslant 3$ we have the following three possibilities.

Case 1 If $n \equiv 0 \ (mod\ 3)$ then the cycle C_n is itself a super Fibonacci graceful.

Case 2 If $n \equiv 1 \ (mod\ 3)$ then $n = 3m+1$ for some positive integer m. Consider the graph G obtained from C_{3m+1} by adding an edge v_0v_{3m-1}. Then the number of edges of G is $|E| = q = 3m+2$. Define $f : V(G) \to \{0, F_1, F_2, \cdots, F_q\}$ as $f(v_0) = 0$ and for $i = 1, 2, 3, \cdots, 3m$,

$$f(v_i) = \begin{cases} F_i & \text{if } i \equiv 1 \ (mod\ 3); \\ F_{i+1} & \text{if } i \equiv 2 \ (mod\ 3); \\ F_{i+2} & \text{if } i \equiv 0 \ (mod\ 3). \end{cases}$$

So for $i \in \{1, 2, 3, \cdots, 3m\}$

$$f(v_{i-1}v_i) = \begin{cases} |F_{i+1} - F_i| & \text{if } i \equiv 1 \ (mod\ 3); \\ |F_{i-1} - F_{i+1}| & \text{if } i \equiv 2 \ (mod\ 3); \\ |F_i - F_{i+2}| & \text{if } i \equiv 0 \ (mod\ 3). \end{cases}$$

Thus

$$f(v_{i-1}v_i) = \begin{cases} F_{i-1} & \text{if } i \equiv 1 \ (mod\ 3); \\ F_i & \text{if } i \equiv 2 \ (mod\ 3); \\ F_{i+1} & \text{if } i \equiv 0 \ (mod\ 3). \end{cases}$$

Also $f(v_0v_{3m}) = |0 - F_{3m+2}| = F_{3m+2}$ and $f(v_0v_{3m-1}) = |F_{3m} - 0| = F_{3m}$. Here each vertex label is either zero or a Fibonacci number at the most F_q and each edge label is also a Fibonacci number at the most F_q. In view of the above defined labeling pattern f admits a super Fibonacci graceful labeling for graph G. That is, G is a super Fibonacci graceful graph.

Case 3 If $n \equiv 2 \pmod 3$ then $n = 3m + 2$ for some positive integer m. Consider the graph G obtained from C_{3m+2} by adding an edge v_0v_{3m-1} and one more edge $v_{3m}v_{3m+2}$ incident to the vertex v_{3m} and a new vertex v_{3m+2}. Then the number of edges of G is $|E| = q = 3m + 4$. Define $f : V(G) \to \{0, 1, 2, \cdots, F_q\}$ as $f(v_0) = 0$ and for $i = 1, 2, 3, \cdots, 3m$,

$$f(v_i) = \begin{cases} F_i & \text{if } i \equiv 1 \pmod 3; \\ F_{i+1} & \text{if } i \equiv 2 \pmod 3; \\ F_{i+2} & \text{if } i \equiv 0 \pmod 3. \end{cases}$$

also $f(v_{3m+1}) = F_{3m+4}$, $f(v_{3m+2}) = 2F_{3m+2}$. So for $i \in \{1, 2, 3, \cdots, 3m\}$ we get that

$$f(v_{i-1}v_i) = \begin{cases} |F_{i+1} - F_i| & \text{if } i \equiv 1 \pmod 3; \\ |F_{i-1} - F_{i+1}| & \text{if } i \equiv 2 \pmod 3; \\ |F_i - F_{i+2}| & \text{if } i \equiv 0 \pmod 3. \end{cases}$$

$$f(v_{i-1}v_i) = \begin{cases} F_{i-1} & \text{if } i \equiv 1 \pmod 3; \\ F_i & \text{if } i \equiv 2 \pmod 3; \\ F_{i+1} & \text{if } i \equiv 0 \pmod 3. \end{cases}$$

Also $f(v_0v_{3m+1}) = |F_{3m+4} - 0| = F_{3m+4} = F_q$, $f(v_{3m}v_{3m-1}) = |F_{3m+2} - F_{3m}| = F_{3m+1}$ and $f(v_{3m}v_{3m+2}) = |F_{3m+2} - 2F_{3m+2}| = F_{3m+2}$.

In view of the above defined labeling pattern f admits a Fibonacci graceful labeling for graph G. That is, G is a Fibonacci graceful graph. □

Remark 3.4 In Case 3, if $n \equiv 2 \pmod 3$ then $f(v_{3m+2}) = 2F_{3m+2}$ which is not a Fibonacci number. Therefore such embedding is not a super Fibonacci graceful. Thus to embed a cycle C_n with $n \equiv 2 \pmod 3$ as a subgraph of a super Fibonacci graceful graph remains an open problem.

Illustration 3.5 A super Fibonacci graceful embedding of the cycle C_7 is shown in Fig.7.

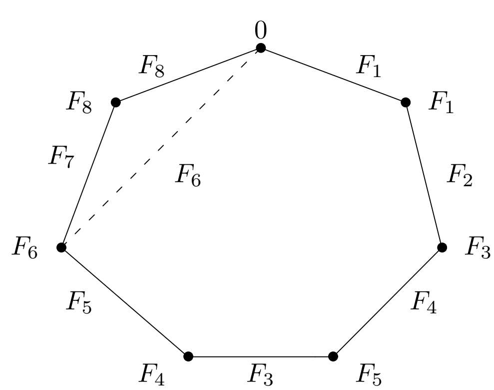

Fig.7

Illustration 3.6 A Fibonacci graceful embedding of the cycle C_8 is shown in Fig.8.

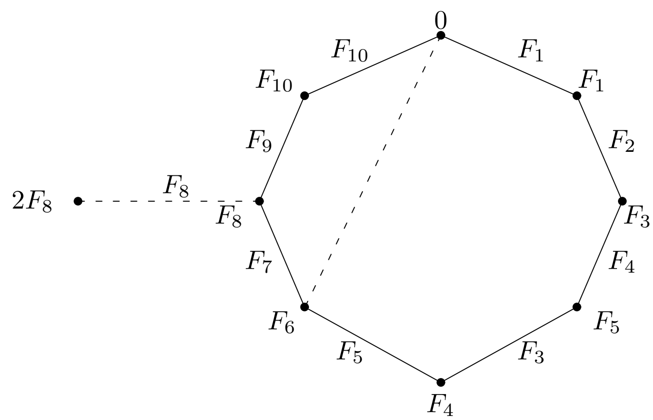

Fig.8

Theorem 3.7 *One point union of k cycles C_n (where $n \equiv 0 \pmod 3$) is a super Fibonacci graceful graph.*

Proof Let the graph G be obtained by taking one point union of k cycles $C^i_{3n_i}$ of order $3n_i$ for each $i = 0, 1, 2, 3 \cdots, k-1$. Let the vertices of each of the cycles $C^i_{3n_i}$ be $u_{i,0}, u_{i,1}, \cdots, u_{i,3n_i-1}$ for each $i = 0, 1, 2, \cdots, k-1$. Let the vertices $u_{0,0}, u_{1,0}, \cdots, u_{k-1,0}$ be identifying to a vertex u_0. So the number of edges $|E| = q = 3(n_0 + n_1 + n_2 + \cdots + n_{k-1})$.

Define $f : V \longrightarrow \{0, 1, 2, 3, \cdots, F_q\}$ as follows:

$f(u_0) = 0$; for $i = 1, 2, \cdots, 3n_0 - 1$,

$$f(u_{0,i}) = \begin{cases} F_i & \text{if } i \equiv 1 \pmod 3; \\ F_{i+1} & \text{if } i \equiv 2 \pmod 3; \\ F_{i+2} & \text{if } i \equiv 0 \pmod 3; \end{cases}$$

and for each $j = 1, 2, 3, \cdots, k-1$,

$$f(u_{j,i}) = \begin{cases} F_{(3\sum_{t=0}^{j-1} n_t + i)} & \text{if } i \equiv 1 \pmod 3; \\ F_{(3\sum_{t=0}^{j-1} n_t + i + 1)} & \text{if } i \equiv 2 \pmod 3; \\ F_{(3\sum_{t=0}^{j-1} n_t + i + 2)} & \text{if } i \equiv 0 \pmod 3. \end{cases}$$

In view of the above defined labeling pattern f admits a super Fibonacci graceful labeling of the graph G. That is, G is a super Fibonacci graceful graph. $\square$

Illustration 3.8 A super Fibonacci graceful labeling of the one point union of three cycles C_3, C_6 and C_3 is as shown in Fig.9.

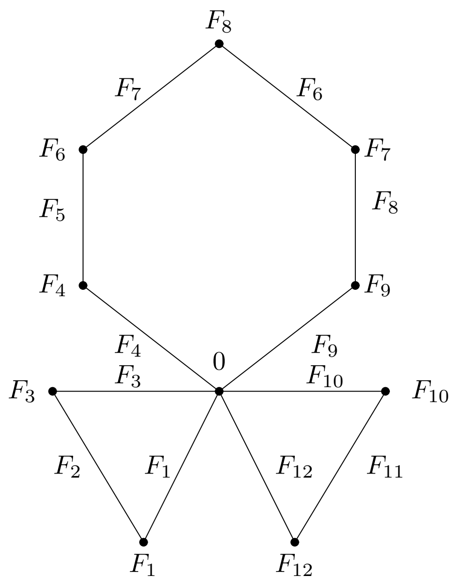

Fig.9

§4. **Concluding Remarks**

Here we investigate four new results corresponding to Fibonacci graceful labeling and three new results corresponding to super Fibonacci graceful labeling of graphs. Analogous results can be derived for other graph families and in the context of different graph labeling problems.

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New Version of Spacelike Horizontal Biharmonic Curves with Timelike Binormal According to Flat Metric in Lorentzian Heisenbegr Group Heis^3

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Abstract: In this paper, we study spacelike biharmonic curves with timelike binormal according to flat metric in the Lorentzian Heisenberg group Heis^3. We determine the parametric representation of the spacelike horizontal biharmonic curves with timelike binormal according to flat metric.

Key Words: Biharmonic curve, Heisenberg group, Flat metric.

AMS(2010): 31B30, 58E20

§1. Introduction

Let (N,h) and (M,g) be Riemannian manifolds. A smooth map $\phi : N \longrightarrow M$ is said to be *biharmonic* if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} \left|\mathcal{T}(\phi)\right|^2 dv_h,$$

where the section $\mathcal{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ.

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_\phi \mathcal{T}(\phi) + \mathrm{tr}R\left(\mathcal{T}(\phi), d\phi\right) d\phi, \tag{1.1}$$

and called the *bitension field* of ϕ. Obviously, every harmonic map is biharmonic. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this paper, we study spacelike biharmonic curves with timelike binormal according to flat metric in the Lorentzian Heisenberg group Heis^3. We determine the parametric representation of the spacelike horizontal biharmonic curves with timelike binormal according to flat metric.

§2. The Lorentzian Heisenberg Group Heis^3

The Heisenberg group Heis^3 is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined as

$$(x, y, z) * (\overline{x}, \overline{y}, \overline{z}) = (x + \overline{x}, y + \overline{y}, z + \overline{z} - \overline{x}y + x\overline{y}).$$

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The identity of the group is $(0,0,0)$ and the inverse of (x,y,z) is given by $(-x,-y,-z)$. The left-invariant Lorentz metric on Heis^3 is

$$g = dx^2 + (xdy + dz)^2 - ((1-x)\,dy - dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{\mathbf{e}_1 = \frac{\partial}{\partial x},\ \mathbf{e}_2 = \frac{\partial}{\partial y} + (1-x)\,\frac{\partial}{\partial z},\ \mathbf{e}_3 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\right\}. \tag{2.1}$$

The characteristic properties of this algebra are the following commutation relations:

$$[\mathbf{e}_2, \mathbf{e}_3] = 0, \quad [\mathbf{e}_3, \mathbf{e}_1] = \mathbf{e}_2 - \mathbf{e}_3, \quad [\mathbf{e}_2, \mathbf{e}_1] = \mathbf{e}_2 - \mathbf{e}_3,$$

with

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = 1, \quad g(\mathbf{e}_3, \mathbf{e}_3) = -1. \tag{2.2}$$

Proposition 2.1 . *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{e}_2 - \mathbf{e}_3 & -\mathbf{e}_1 & -\mathbf{e}_1 \\ \mathbf{e}_2 - \mathbf{e}_3 & -\mathbf{e}_1 & -\mathbf{e}_1 \end{pmatrix}, \tag{2.3}$$

where the (i,j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

So we obtain that

$$R(\mathbf{e}_1, \mathbf{e}_3) = R(\mathbf{e}_1, \mathbf{e}_2) = R(\mathbf{e}_2, \mathbf{e}_3) = 0. \tag{2.4}$$

Then, the Lorentz metric g is flat.

§3. Spacelike Horizontal Biharmonic Curves with Timelike Binormal According to Flat Metric in the Lorentzian Heisenberg Group Heis^3

An arbitrary curve $\gamma : I \longrightarrow Heis^3$ is spacelike, timelike or null, if all of its velocity vectors $\gamma'(s)$ are, respectively, spacelike, timelike or null, for each $s \in I \subset \mathbb{R}$. Let $\gamma : I \longrightarrow Heis^3$ be a unit speed spacelike curve with timelike binormal and $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ are Frenet vector fields, then Frenet formulas are as follows

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= \kappa_1 \mathbf{n}, \\ \nabla_{\mathbf{t}}\mathbf{n} &= -\kappa_1 \mathbf{t} + \kappa_2 \mathbf{b}, \\ \nabla_{\mathbf{t}}\mathbf{b} &= \kappa_2 \mathbf{n}, \end{aligned} \tag{3.1}$$

where κ_1, κ_2 are curvature function and torsion function, respectively and

$$\begin{aligned} g(\mathbf{t},\mathbf{t}) &= 1,\ g(\mathbf{n},\mathbf{n}) = 1,\ g(\mathbf{b},\mathbf{b}) = -1, \\ g(\mathbf{t},\mathbf{n}) &= g(\mathbf{t},\mathbf{b}) = g(\mathbf{n},\mathbf{b}) = 0. \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{t} &= t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3, \\ \mathbf{n} &= n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3, \\ \mathbf{b} &= b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3. \end{aligned}$$

Theorem 3.1 *If $\gamma : I \longrightarrow Heis^3$ is a unit speed spacelike biharmonic curve with timelike binormal according to flat metric, then*

$$\begin{aligned} &\kappa_1 = \text{constant} \neq 0, \\ &\kappa_1^2 - \kappa_2^2 = 0, \\ &\kappa_2 = \text{constant}. \end{aligned} \tag{3.2}$$

Lemma 3.2 *If $\gamma : I \longrightarrow Heis^3$ is a unit speed spacelike biharmonic curve with timelike binormal, then γ is a helix.*

Theorem 3.3 *Let $\gamma : I \longrightarrow Heis^3$ is a unit speed spacelike biharmonic curve with timelike binormal according to flat metric. Then the parametric equations of γ are*

$$\begin{aligned} x(s) &= \frac{\cosh^2\varphi}{\kappa_1}\sin\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right] + C_1, \\ y(s) &= -\frac{\cosh^2\varphi}{\kappa_1}\cos\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right] + s\sinh\varphi + C_2, \\ z(s) &= -\frac{\cosh^3\varphi}{\kappa_1}\left(\frac{s}{2} - \frac{\cosh\varphi}{\kappa_1}\sin 2\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right]\right) \\ &\quad -\frac{1}{\kappa_1}\left(\cosh^2\varphi - \frac{\sinh\varphi\cosh^3\varphi}{\kappa_1}\right)\cos\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right] + C_3, \end{aligned} \tag{3.3}$$

where C_1, C_2, C_3 are constants of integration.

Proof Assume that γ is a unit speed spacelike biharmonic curve with timelike binormal according to flat metric in the Lorentzian Heisenberg group Heis^3. Using Lemma 3.2 without loss of generality, we take the axis of γ is parallel to the spacelike vector $\mathbf{e}_3$. Then,

$$g(\mathbf{t}, \mathbf{e}_3) = t_3 = \sinh\varphi, \tag{3.4}$$

where φ is constant angle.

Direct computations show that

$$\mathbf{t} = \cosh\varphi\cos\mathbb{k}\mathbf{e}_1 + \cosh\varphi\sin\mathbb{k}\mathbf{e}_2 + \sinh\varphi\mathbf{e}_3. \tag{3.5}$$

Using above equation and Frenet equations, we obtain

$$\mathbb{k} = \frac{\kappa_1 s}{\cosh\varphi} + \aleph, \tag{3.6}$$

where $\aleph$ is a constant of integration.

From these we get the following formula

$$\mathbf{t} = \cosh\varphi\cos\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right]\mathbf{e}_1 + \cosh\varphi\sin\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right]\mathbf{e}_2 + \sinh\varphi\mathbf{e}_3. \tag{3.7}$$

Therefore, Equation (3.9) becomes

$$\begin{aligned}\mathbf{t} = (\cosh\varphi\cos\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right], \cosh\varphi\sin\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right] + \sinh\varphi,\\ (1-x)\cosh\varphi\sin\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right] - x\sinh\varphi).\end{aligned} \tag{3.8}$$

Now using Equation (3.10) we obtain

$$\begin{aligned}\frac{dx}{ds} &= \cosh\varphi\cos\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right],\\ \frac{dy}{ds} &= \cosh\varphi\sin\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right] + \sinh\varphi,\\ \frac{dz}{ds} &= -\frac{\cosh^3\varphi}{\kappa_1}\sin^2\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right]\\ &\quad +(\cosh\varphi - \frac{\sinh\varphi\cosh^2\varphi}{\kappa_1})\sin\left[\frac{\kappa_1 s}{\cosh\varphi} + \aleph\right].\end{aligned} \tag{3.9}$$

With direct computations on above system we have Equation (3.3). The proof is completed.□

Using Mathematica in above Theorem, we have following figure.

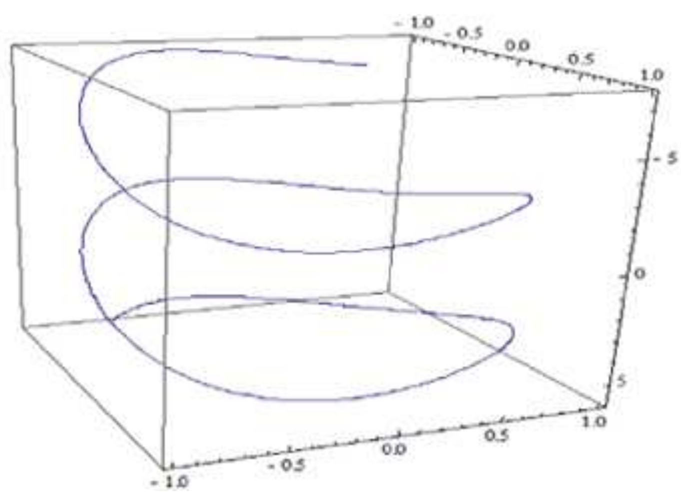

Fig.1

Theorem 3.4 *Let $\gamma : I \longrightarrow Heis^3$ is a unit speed spacelike horizontal biharmonic curve with timelike binormal according to flat metric. Then the parametric equations of γ are*

$$\begin{aligned} x(s) &= \frac{1}{\kappa_1}\sin[\kappa_1 s + \aleph] + C_1, \\ y(s) &= -\frac{1}{\kappa_1}\cos[\kappa_1 s + \aleph] + C_2, \\ z(s) &= -\frac{1}{\kappa_1}(\frac{s}{2} - \frac{1}{\kappa_1}\sin 2[\kappa_1 s + \aleph]) - \frac{1}{\kappa_1}\cos[\kappa_1 s + \aleph] + C_3, \end{aligned}$$

where C_1, C_2, C_3 are constants of integration.

Corollary 3.5 *If $\gamma : I \longrightarrow Heis^3$ is a unit speed spacelike biharmonic curve with timelike binormal according to flat metric. Then*

$$\kappa_1 = \mp\kappa_2. \tag{3.10}$$

Theorem 3.6 *Let $\gamma : I \longrightarrow Heis^3$ is a unit speed spacelike horizontal biharmonic curve with timelike binormal according to flat metric. Then the parametric equations of γ in terms of torsion are*

$$\begin{aligned} x(s) &= \mp\frac{1}{\kappa_2}\sin[\mp\kappa_2 s + \aleph] + C_1, \\ y(s) &= \mp\frac{1}{\kappa_2}\cos[\mp\kappa_2 s + \aleph] + C_2, \\ z(s) &= \mp\frac{1}{\kappa_2}(\frac{s}{2} \mp \frac{1}{\kappa_2}\sin 2[\mp\kappa_2 s + \aleph]) \mp \frac{1}{\kappa_2}\cos[\mp\kappa_2 s + \aleph] + C_3, \end{aligned} \tag{3.11}$$

where C_1, C_2, C_3 are constants of integration.

Proof Using Equation (3.10) in Equation (3.3), we obtain Equation (3.11). Thus, the proof is completed. □

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On The Isoperimetric Number of Line Graphs

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Abstract: The *isoperimetric number* of a graph G, denoted $i(G)$, was introduced in 1987 by Mohar [8]. Given a graph G and a subset of X of its vertices, let $\partial(X)$ denote the edge boundary of X: i.e. the set of edges which connect vertices in X with vertices not in X. The isoperimetric number of G defined as $i(G) = min_{1 \leq |X| \leq \frac{|V(G)|}{2}} \frac{|\partial(X)|}{|X|}$. This paper obtains some results about the isoperimetric number of graphs obtained from graph operations are given.

Key Words: Isoperimetric number, line graph, graph operations.

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§1. Introduction

The *isoperimetric number* of a graph G, denoted $i(G)$, was introduced in 1987 by Mohar [8]. Given a graph G and a subset of X of its vertices , let $\partial(X)$ denote the edge boundary of X, i.e. the set of edges which connect vertices in X with vertices not in X. The isoperimetric number of G defined as

$$i(G) = min_{1 \leq |X| \leq \frac{|V(G)|}{2}} \frac{|\partial(X)|}{|X|}. \tag{1.1}$$

Clearly, $i(G)$ can be defined in a more symmetric form as

$$i(G) = min \frac{|E(X,Y)|}{min\{|X|,|Y|\}}, \tag{1.2}$$

where the minimum runs over all partitions of $V(G) = X \cup Y$ into non empty subsets X and Y , and $E(X,Y) = \partial X = \partial Y$ are the edges between X and Y.

The importance of *i(G)* lies in various interesting interpretations of this number [8]:

(1) From (1.2) it is evident that, in trying to determine *i(G)*, we have to find a small edge-cut *E(X,Y)* separating as large a subset X (assume $|X| \leq |Y|$) as possible from the remaining larger part Y. So, it is evident that *i(G)* can serve as measure of *connectivity* of graphs. It seems that there might be possible applications in problems concerning connected networks and

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the ways to "destroy" them by removing a large portion of the network by cutting only a few edges.

(2) The problem of the partitioning *V(G)* into two equally sized subsets (to within one element) in such a way that the number of the edges in the cut is minimal, is known as the *bisection width* problem. It is important in VLSI design and some other practical applications. Clearly, it is related to isoperimetric number.

In Section 2 known results on the isoperimetric number and some definitions are given. Section 3 gives some results about the isoperimetric number of graphs obtained from graph operations.

§2. **Basic Results**

In this section we will review some of the known results.

Theorem 2.1([8]) *Let m, n be the positive integers. The isoperimetric number of some common graphs are as follows,*

(1) *The complete graph K_n has $i(K_n) = \lceil \frac{n}{2} \rceil$;*

(2) *The cycle C_n has $i(C_n) = \frac{2}{\lfloor \frac{n}{2} \rfloor}$;*

(3) *The path P_n on n vertices has $i(P_n) = \frac{1}{\lfloor \frac{n}{2} \rfloor}$;*

(4) *The isoperimetric numbers of complete bipartite graphs $K_{m,n}$ are respectively*

$$i(K_{m,n}) = \begin{cases} \dfrac{mn}{m+n} & if\ m\ and\ n\ are\ even, \\ \dfrac{mn+1}{m+n} & if\ m\ and\ n\ are\ odd, \\ \dfrac{mn}{m+n-1} & if\ m+n\ is\ odd, \end{cases}$$

it can be shortened to $i(K_{m,n}) = \lceil mn/2 \rceil / \lfloor (m+n)/2 \rfloor$.

Theorem 2.2([8]) *Some of the theorems that Mohar state are below,*

(1) *$i(G) = 0$ if and only if G is disconnected;*

(2) *If G is k-edge-connected then $i(G) \geq 2k/|V(G)|$;*

(3) *If δ is the minimal degree of vertices in G then $i(G) \leq \delta$;*

(4) *If $e = uv$ is an edge of G and $|V(G)| \geq 4$ then*

$$i(G) \leq \frac{deg(u) + deg(v) - 2}{2};$$

(5) *If Δ is the maximum vertex degree in G then $i(G) \leq (\Delta - 2) + 2/\lfloor |V(G)|/2 \rfloor$. If G has cycle with a most half the vertices of G then $i(G) \leq \Delta - 2$.*

Now we will give some definitions.

Definition 1.1([7]) *The line graph of G denoted $L(G)$, is the intersection graph $\Omega(x)$. Thus the points of $L(G)$ are the lines of G with two points of $L(G)$ adjacent whenever the corresponding lines of G are. If $x = uv$ is a line of G, then the degree of x in $L(G)$ is clearly $deg(u)+deg(v)-2$.*

Definition 1.2([7]) *A subset S of $V(G)$ such that every edge of G has at least one end in S called a covering set of G. The number of vertices in a minimum covering set of G is the covering number of G is denoted by $\alpha(G)$.*

§3. Isoperimetric Number of Line Graphs

Firstly, we can say the following observation for the isoperimetric number of line graphs of graphs P_n and C_n.

- P_n – Let P_n be a path graph with n vertices. Then $i(L(P_n)) = i(P_{n-1})$.
- C_n – Let C_n be a cycle graph with n vertices. Then $i(L(C_n)) = i(C_n)$.
- If G is a graph with $\alpha(G) = 1$, then $i(L(G)) = \lceil \frac{n}{2} \rceil$.

Next we consider some of the operations on graphs. We start with the complement of a graph and give the definition its.

Definition 3.1([7]) *The complement of $\overline{G}$ of a graph G is the graph with vertex set $V(G)$ defined by the edge $e \in E(\overline{G})$ if only if $e \notin E(G)$, where $e = uv, u, v, \in V(G)$.*

Theorem 3.1 *Let $\overline{L(P_n)}$ be a complement of line graph of path graph with $n-1$ vertices then*

$$i(\overline{L(P_n)}) = \begin{cases} \frac{n}{2} - 2, & if\ n\ is\ even, \\ \dfrac{(\frac{n-1}{2} - 1)^2}{\frac{n-1}{2}}, & if\ n\ is\ odd, \end{cases}$$

Proof First, we prove this result for odd n. Let $X \subseteq V(\overline{L(P_n)})$ where $|X| \leq \frac{n-1}{2}$ and let $V(\overline{L(P_n)}) = 1, 2, \ldots, n-1$. Assume that $X \subseteq V(\overline{L(P_n)})$ and $V(\overline{L(P_n)}) = V(G_1) \cup V(G_2)$ where $V(G_1) = 1, 3, \ldots, n-2$ and $V(G_2) = 2, 4, \ldots, n-1$. $(\overline{L(P_n)})$ contains two complete graphs $K_{\frac{n-1}{2}}$ formed by the vertices of $V(G_1)$ and $V(G_2)$ respectively.

Case 1 Assume that $|X| \subseteq V(G_1)$ and $|X \cap V(G_2)| = 0$ or $|X| \subseteq V(G_2)$ and $|X \cap V(G_1)| = 0$ where $|X| < \frac{n-1}{2}$. If $|X| = r$ then $|\partial(X)| \geq r(\frac{n-1}{2} - 2)$. Therefore

$$\frac{|\partial(X)|}{|X|} \geq \frac{r(\frac{n-1}{2} - 2)}{r} = (\frac{n-1}{2} - 2).$$

Case 2 Suppose that $|X| \subseteq V(G_1)$, $|X \cap V(G_2)| = 0$ and $|X| = \frac{n-1}{2}$ or $|X| \subseteq V(G_2)$, $|X \cap V(G_1)| = 0$ and $|X| = \frac{n-1}{2}$. Since $|X| = r$ then $|\partial(X)| = (\frac{n-1}{2} - 2)(\frac{n-1}{2} - 1) +$

$(\frac{n-1}{2}-1)=(\frac{n-1}{2}-1)^2$. Therefore

$$\frac{|\partial(X)|}{|X|}=\frac{(\frac{n-1}{2}-1)^2}{\frac{n-1}{2}}.$$

Case 3 Let $X\subseteq V(\overline{L(P_n)})$ where $|X\cap V(G_1)|=a$ and $|X\cap V(G_2)|=b$ for $a\neq 0$ and $b\neq 0$. We have two cases according to $(a+b)$.

Subcase 1 Let $a+b<\frac{n-1}{2}$. In this case a vertices are connected to G_2 with $(\frac{n-1}{2}-2-b)$ edges and connected $(|V(G_1)|-a)$ with $(\frac{n-1}{2}-a)$ edges. Similarly b vertices are connected to G_1 with $(\frac{n-1}{2}-2-a)$ edges and connected $(|V(G_2)|-b)$ with $(\frac{n-1}{2}-a)$ edges. Hence

$$\begin{aligned}|\partial(X)| &\geq a(\frac{n-1}{2}-2-b)+b(\frac{n-1}{2}-2-a)+a(\frac{n-1}{2}-a)+b(\frac{n-1}{2}-a)\\ &= (a+b)(n-1-a-b-2).\end{aligned}$$

Therefore

$$\frac{|\partial(X)|}{|X|}\geq\frac{(a+b)(n-1-a-b-2)}{a+b}\geq n-1-(a+b)-2\geq\frac{n-1}{2}-1.$$

Subcase 2 If $a+b=\frac{n-1}{2}$ then

$$\begin{aligned}|\partial(X)| &\geq a(\frac{n-1}{2}-2-b)+b(\frac{n-1}{2}-2-a)+a(\frac{n-1}{2}-a)+b(\frac{n-1}{2}-a)+1+1\\ &= (a+b)(n-1-a-b-2)+2.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{|\partial(X)|}{|X|} &\geq \frac{(a+b)(n-1-a-b-2)+2}{a+b}\\ &\geq n-1-(a+b)-\frac{2}{a+b}\\ &\geq \frac{n-1}{2}-2+\frac{2}{n-1}.\end{aligned}$$

Combining Cases $1-3$, the proof is completed for odd n.

For the case of n being even, the proof is very similar to that of odd n. □

Theorem 3.2 *Let $\overline{L(C_n)}$ be a complement of line graph of cycle graph with n vertices then*

$$i(\overline{L(C_n)})=\begin{cases}\frac{n}{2}-2, & if\ n\ is\ even,\\ \frac{n+1}{2}-2, & if\ n\ is\ odd.\end{cases}$$

Proof The proof is similar to that of Theorem 3.1. □

We consider now the isoperimetric number of the join of two graphs and give the definition of join operation.

Definition 3.2([7]) *Let G_1 and G_2 be two graphs. The union $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. The join is denoted $V(G_1) + V(G_2)$ and consists of $V(G_1) \cup V(G_2)$ and all edges joining $V(G_1)$ with $V(G_2)$.*

Let us first consider the join of the graph K_1 with cycle C_n.

Theorem 3.3 *Let $K_1 + C_n$ be a graph with $n+1$ vertices then*

$$i(L(K_1 + C_n)) = 2.$$

Proof The graph $K_1 + C_n$ consists of edges of cycle graph C_n and edges of star graph $K_{1,n}$. Let $E(C_n) = \{e_1, e_2, \ldots, e_n\}$ and $E(K_{1,n}) = \{h_1, h_2, \ldots, h_n\}$. Assume that $V(K_1 + C_n) = V(G_1) \cup V(G_2)$ such that $V(G_1) = \{e_1, e_2, \ldots, e_n\}$ and $V(G_2) = \{h_1, h_2, \ldots, h_n\}$. Let $X \subseteq V(L(K_1 + C_n))$ with $|X| \leq n$ and $|X \cap V(G_1)| = a, |X \cap V(G_2)| = b$. Therefore there are have five cases according to a and b.

Case 1 If $a = 0$, $b \leq n$ and $|X| = b$ then $|\partial(X)| = b.(n-b) + 2b$. Hence

$$\frac{|\partial(X)|}{|X|} = \frac{b(n-b)+2b}{b}.$$

The function $n - b + 2$ takes its minimum value at $b = n$ and $i(L(K_1 + C_n)) = 2$.

Case 2 If $a = b$ and $|X| = a + b$ then $|\partial(X)| = 2 + b.(n-b) + 2$. Thus

$$\frac{|\partial(X)|}{|X|} = \frac{2+b(n-b)+2}{a+b} = \frac{2+b(n-b)+2}{2b}.$$

The function $\frac{2+b(n-b)+2}{2b}$ takes its minimum value at $a = b = \lfloor \frac{n}{2} \rfloor$ and we have

$$i(L(K_1 + C_n)) = \frac{2 + \lfloor \frac{n}{2} \rfloor (n - \lfloor \frac{n}{2} \rfloor) + 2}{2 \lfloor \frac{n}{2} \rfloor}.$$

Case 3 If $a \neq b$, $0 < a < n$, $0 \leq b < n$, $a < b$ and $|X| = a + b$, then $|\partial(X)| = 2 + b(n-b) + (b-a) \times 2$. Therefore we get

$$\begin{aligned} \frac{|\partial(X)|}{|X|} &= \frac{2+b(n-b)+2(b-a)}{a+b} \geq \frac{2+b(n-b)+2}{2b} \\ &\geq \frac{4+ba}{2b} \geq 2. \end{aligned}$$

Case 4 If $a \neq b$, $0 < a < n$, $0 \leq b < n$, $a > b$ and $|X| = a + b$, then $|\partial(X)| = 2 + b(n-b) + 2(a-b)$. Thus

$$\begin{aligned} \frac{|\partial(X)|}{|X|} &= \frac{2+b(n-b)+2(a-b)}{a+b} \geq \frac{2+b(n-b)+2}{2b} \\ &\geq \frac{4+ba}{2b} \geq 2 \end{aligned}$$

Case 5 If $a = n$, $b = 0$ and $|X| = n$ then $|\partial(X)| = 2n$. Hence $\frac{|\partial(X)|}{|X|} = 2$.

Combining Cases $1 - 4$, the proof is completed. □

Theorem 3.4 *Let $K_1 + P_n$ be a graph with $n+1$ vertices then*

$$i(L(K_1 + P_n)) = 2.$$

Proof The proof is similar to that of Theorem 3.3. □

Finally we consider the cartesian product of two graphs.

Definition 3.3([7]) *The Cartesian product $G_1 \times G_2$ of graphs G_1 and G_2 has $V(G_1) \times V(G_2)$ as its vertex set and (u_1, u_2) is adjacent to (v_1, v_2) if either $u_1 = v_1$ and u_2 is adjacent to v_2 or $u_2 = v_2$ and u_1 is adjacent to v_1.*

Theorem 3.5 following is given by Mohar in [8].

Theorem 3.5([8]) *If G is a graph having an even number number of vertices for every $n \geq 1$,*

$$i(K_{2n} \times G) = min\{i(G), n\}.$$

By applying Theorem 3.5, we can easily get the following observation.

• Let $K_2 \times P_n$ be the cartesian product of K_2 and P_n be a graph with $2n$ vertices. Then $i(K_2 \times P_n) = i(P_n)$.

• Let $K_2 \times C_n$ be the cartesian product of K_2 and C_n be a graph with $2n$ vertices. Then $i(K_2 \times C_n) = i(C_n)$.

The following theorems give the isoperimetric number of graphs $L(K_2 \times P_n)$ and $L(K_2 \times C_n)$.

Theorem 3.6 *Let $K_2 \times P_n$ be the cartesian product of K_2 and P_n be a graph with $2n$ vertices. Then*

$$i(L(K_2 \times P_n)) = \begin{cases} \dfrac{8}{3n-2} & if\ n\ is\ even, \\ \dfrac{8}{3n-3} & if\ n\ is\ odd. \end{cases}$$

Proof Let P_1 and P_2 be two path graphs contained in $K_2 \times P_n$. Assume that P_1 has a vertex labelling through $v_1 - v_n$ such that $V(P_1) = \{v_i | v_i$ is adjacent to v_{i+1} for $1 \leq i \leq n\}$. Similarly P_2 has a labelling through $u_1 - u_n$ such that $V(P_2) = \{u_j | u_j$ is adjacent to v_{j+1} for $1 \leq i \leq n\}$.

$L(K_2 \times P_n)$ has $3n-2$ vertices and let $V(L(K_2 \times P_n)) = V(G_1) \cup V(G_2) \cup V(G_3)$. Suppose that the edges along the path $v_1 - v_n$ which form the vertices of G_1 have a labelling such that $V(G_1) = \{e_i | e_i$ is adjacent to e_{i+1} for $1 \leq i \leq n-1\}$. Similarly assume that the edges along the path $u_1 - u_n$ which form the vertices of G_1 have a labelling such that $V(G_2) = \{m_i | m_i$ is adjacent to m_{i+1} for $1 \leq i \leq n-1\}$. In addition suppose the vertices of G_3 have a labelling such that $V(G_3) = \{k_i | i = j$ for $1 \leq i \leq n$ and $1 \leq j \leq n$ where v_i and u_j are the vertices of P_1 and P_2 respectively $\}$.

There are three cases according to the cardinality of X where $X \subset V(L(K_2 \times P_n))$ and $|X| \leq \lfloor \frac{3n-2}{2} \rfloor$.

Case 1 If $|X| = 1$ then $\delta = 2$ and $i(L(K_2 \times P_n)) = 2$.

Case 2 If $|X| = 2$ then we have $|\partial(X)| \geq 3$. Then

$$\frac{|\partial(X)|}{|X|} \geq \frac{3}{2}.$$

Case 3 Let $|X| > 2$. The discussion is divided into subcases following.

Subcase 1 Let n be even and $|X| = r$. If $2 < r \leq \frac{3n-2}{2}$, then we have $|\partial(X)| \geq 4$. Hence we have $\frac{|\partial(X)|}{|X|} \geq \frac{4}{r}$. The function $\frac{4}{r}$ takes its minimum value at $r = \frac{3n-2}{2}$ and we get

$$\frac{|\partial(X)|}{|X|} \geq \frac{4}{\frac{3n-2}{2}}.$$

It can be easily seen that there exists a set X such that $X = \{e_1, e_2, \ldots, e_{\frac{n}{2}}, k_1, k_2, \ldots, k_{\frac{n}{2}}, m_1, m_2, \ldots, m_{\frac{n}{2}-1}\}$ and we have $|\partial(X)| = 4$. Therefore, we get

$$i(L(K_2 \times P_n)) = \frac{4}{\frac{3n-2}{2}}.$$

Whence, $i(L(K_2 \times P_n)) = \frac{8}{3n-2}$ for n even.

Subcase 2 Let n be odd and $|X| = r$. If $2 < r \leq \frac{3n-3}{2}$, then we have $|\partial(X)| \geq 4$. Hence we have $\frac{|\partial(X)|}{|X|} \geq \frac{4}{r}$. The function $\frac{4}{r}$ takes its minimum value at $r = \frac{3n-3}{2}$ and we get that

$$\frac{|\partial(X)|}{|X|} \geq \frac{4}{\frac{3n-3}{2}}.$$

It can be easily seen that there exists a set X such that $X = \{e_1, e_2, \ldots, e_{\frac{n-1}{2}}, k_1, k_2, \ldots, k_{\frac{n-1}{2}}, m_1, m_2, \ldots, m_{\frac{n-1}{2}}\}$ and we have $|\partial(X)| = 4$. Therefore we get $i(L(K_2 \times P_n)) = \frac{4}{\frac{3n-3}{2}}$. Therefore, $i(L(K_2 \times P_n)) = \frac{8}{3n-3}$ for n odd. □

Theorem 3.7 *Let $K_2 \times C_n$ be the cartesian product of K_2 and P_n be a graph with $2n$ vertices. Then*

$$i(L(K_2 \times C_n)) = \frac{8}{\lceil \frac{3n}{2} \rceil}.$$

Proof The proof is similar to that of Theorem 3.6. □

§4. **Conclusion**

In this paper isoperimetric number of line graphs are studied. Some results for the isoperimetric number of graphs obtained by graph operations such as complement, join operation and cartesian product are obtained. To make further progress in this direction, one could try to characterize the graphs with given isoperimetric number.

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On the Series Expansion of the Ramanujan Cubic Continued Fraction

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Abstract: If the Ramanujan cubic continued fraction (or its reciprocal) is expanded as a power series, the sign of the coefficients is periodic with period 3. We give the combinatorial interpretations for the coefficients from which the result follows immediate. We also derive some interesting identities involving coefficients.

Key Words: Ramanujan cubic continued fraction, Jacobi's triple product identity, partitions, color partitions.

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§1. Introduction

§1. Introduction

As usual for any complex number a, we define

$$(a)_0 := 1,$$
$$(a)_n := (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$$

where q is any complex number with $|q| < 1$. We also define

$$(a)_\infty := (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n),$$
$$(a_1,a_2,a_3,...,a_n;q)_\infty := \prod_{i=1}^{n}(a_i;q)_\infty$$

and

$$(q^{r\pm};q^s)_\infty := (q^r,q^{s-r};q^s)_\infty,$$

where s and r are positive integers with $r < s$. One of the most celebrated q-series identity is

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the following Jacobi's triple product identity:

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = \left(\frac{-q}{z}; q^2\right)_\infty \left(-qz; q^2\right)_\infty \left(q^2; q^2\right)_\infty, \qquad |z| < 1. \tag{1.1}$$

Ramanujan [13], [6] expressed the above identity in the following form:

$$\begin{aligned} f(a,b):&= \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} \\ &= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \qquad |ab| < 1. \end{aligned} \tag{1.2}$$

Further, Ramanujan defined the following particular case of $f(a,b)$:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \tag{1.3}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{1.4}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty.$$

Ramanujan also define

$$\chi(q) := (-q;q^2)_\infty.$$

The famous Rogers-Ramanujan continued fraction is defined as

$$R(q) := 1 + \frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+\ldots}.$$

B. Richmond and G. Szekeres [15] examined asymptotically the power series co-efficients of $R(q)$, in particular if

$$R(q) := \sum_{n=0}^{\infty} r_n q^n,$$

they have proved that for n sufficiently large

$$r_{5n},\ r_{5n+1} > 0$$

and

$$r_{5n+2},\ r_{5n+3},\ r_{5n+4} < 0.$$

A similar result was also shown for the coefficients of $R^{-1}(q)$. In examining Ramanujan's lost notebook, G. E. Andrews discovered some relevant formulae. He [4] then established a combinatorial interpretation of these formulae. M. D. Hirschhorn [11] later gave a simple proofs of these identities using only quintuple product identity.

On page 229 of his notebook [12], Ramanujan recorded interesting continued fraction $H(q)$ defined by

$$
\begin{aligned}
H(q) &:= \frac{1}{1+q}{}_{+}\frac{q^2}{1+q^3}{}_{+}\frac{q^4}{1+q^5}{}_{+\cdots} \\
&= \frac{f(-q,-q^7)}{f(-q^3,-q^5)}.
\end{aligned} \tag{1.5}
$$

Without any knowledge of Ramanujan's work, Gordon [9] and Göllnitz [10] rediscovered and proved (1.5). Richmond and Szekeres [15], Andrews and D. M. Bressoud [5], K. Alladi and B. Gordon [1], Hirschhorn [12], and S- D. Chen and S- S. Huang [8] have studied the periodicity of signs of Taylor series coefficients of the expansion of $H(q)$.

Recently S. H. Chan and H. Yesilyurt [7] shown the periodicity of large number of quotients of certain infinite products. For example, they have deduced Corollary 2.2 below from their general result.

On page 366 of his lost notebook [14] Ramanajan investigated another beautiful continued fraction $G(q)$ defined by

$$
\begin{aligned}
G(q): \quad &= \frac{1}{1}{}_{+}\frac{q+q^2}{1}{}_{+}\frac{q^2+q^4}{1}{}_{+}\frac{q^3+q^6}{1}{}_{+\cdots} \\
&= \frac{f(-q,-q^5)}{f(-q^3,-q^3)}
\end{aligned}
$$

and claimed that there are many results of $G(q)$ which are analogous to results of $R(q)$. Motivated by Ramanujan's claim and the above mentioned works on $R(q)$ and $H(q)$, in this paper, we give the combinatorial interpretation of the co-efficient in the series expansion of $G(q)$ and its reciprocal.

We conclude this introduction by letting

$$G(q) = \sum_{n=0}^{\infty} a_n q^n \tag{1.6}$$

and

$$M(q) = \frac{1}{G(q)} = \sum_{n=0}^{\infty} b_n q^n. \tag{1.7}$$

§2. Combinatorial Interpretations of a_n and b_n

Lemma 2.1 *We have*

$$G(q) = \frac{1}{\varphi(-q^3)} \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2+2n}. \tag{2.1}$$

Proof We have [6, p. 345]

$$2G(q) = \frac{q^{-\frac{1}{3}}}{\varphi(-q^3)} \left[\varphi(-q^3) - \varphi(-q^{\frac{1}{3}})\right].$$

Upon using (1.3) in the above, we obtain

$$2G(q) = \frac{1}{q^{\frac{1}{3}}\varphi(-q^3)}\left[\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2} - \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n^2}{3}}\right]$$

Converting the second series in the right hand side of the above into sum of three series, we deduce that

$$2G(q) = \frac{1}{\varphi(-q^3)}\left[\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2+2n} - \sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2+4n+1}\right].$$

Now replacing n by $-(n+1)$ in the second series, we obtain

$$G(q) = \frac{1}{\varphi(-q^3)}\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2+2n}. \qquad \square$$

Theorem 2.1 *Let a_n be as defined in (1.6). Then*

$$\sum_{n=0}^{\infty} a_{3n}q^n = \frac{1}{(q;q^2)_\infty (q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{9n^2+2n}, \tag{2.2}$$

$$\sum_{n=0}^{\infty} a_{3n+1}q^n = \frac{-1}{(q;q^2)_\infty (q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{9n^2+4n} \tag{2.3}$$

and

$$\sum_{n=0}^{\infty} a_{3n+2}q^n = \frac{-q}{(q;q^2)_\infty (q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{9n^2+8n}. \tag{2.4}$$

Proof Recall that the operator U_3 [3, p. 161], operating on a power series (1.6) is defined by

$$U_3 G(q) := \sum_{n=0}^{\infty} a_{3n}q^n = \frac{1}{3}\sum_{j=0}^{2} G(t^j q^{\frac{1}{3}}),$$

where $t = e^{\frac{2\pi i}{3}}$. Hence for $0 \le k \le 2$, by Lemma 2.1, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} a_{3n+k}q^n &= U_3\left[q^{-k}G(q)\right] \\
&= \frac{1}{3}\sum_{j=0}^{2}(t^j q^{\frac{1}{3}})^{-k} G(t^j q^{\frac{1}{3}}) \\
&= \frac{1}{3\varphi(-q)}\sum_{n=-\infty}^{\infty}(-1)^n \sum_{j=0}^{2} t^{(3n^2+2n-k)j} q^{\frac{3n^2+2n-k}{3}}.
\end{aligned}$$

Now $3n^2+2n-k \equiv 0 \pmod 3$ for $2n \equiv k \pmod 3$. It therefore follows from the above, that

$$\begin{aligned}
\sum_{n=0}^{\infty} a_{3n+k}q^n &= \frac{1}{\varphi(-q)}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{3(6n+4k)^2+4(6n+4k)-4k}{12}} \\
&= \frac{1}{\varphi(-q)}\sum_{n=-\infty}^{\infty}(-1)^n q^{9n^2+12nk+2n+4k^2+k}.
\end{aligned} \tag{2.5}$$

□

The identities (2.2)-(2.4) now follows by setting $k = 0, 1, 2$ respectively in (2.5). In the case (2.3) the index of the summation needs to be changed by replacing n by $n-1$ and then n to $-n$, in the case (2.4) the index of the summation need to be changed n by $n-1$.

Theorem 2.2 *Let $P_s(n)$ denote the number of partitions of n with parts not congruent to 0 (mod 18) and each odd parts having two colours except parts congruent to $\pm s \equiv$ (mod 18). Then*

$$a_{3n} = P_7(n), \tag{2.6}$$

$$a_{3n+1} = -P_5(n) \tag{2.7}$$

and

$$a_{3n+2} = -P_1(n-1). \tag{2.8}$$

Proof From (1.1), we have

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{9n^2} z^n = \left(\frac{q^9}{z}; q^{18}\right)_\infty \left(zq^9; q^{18}\right)_\infty \left(q^{18}; q^{18}\right)_\infty. \tag{2.9}$$

Using (2.9) with $z = q^2$ in (2.2), we have

$$\sum_{n=0}^{\infty} a_{3n} q^n = \frac{(q^7, q^{11}, q^{18}; q^{18})_\infty}{(q; q^2)_\infty (q; q)_\infty}.$$

Clearly right side of the above identity is the generating function for $P_7(n)$. Thus, we have

$$\sum_{n=0}^{\infty} a_{3n} q^n = \sum_{n=0}^{\infty} P_7(n) q^n,$$

which implies (2.6). Similarly by taking $z = q^4$ and $z = q^8$ in (2.9), using them in (2.3) and (2.4), we obtain (2.7) and (2.8) respectively. □

Example 2.1 By using Maple we have been able to find the following series expansion for $G(q)$

$$\begin{aligned} G(q) &= 1 - q + 2q^3 - 2q^4 - q^5 + 4q^6 - 4q^7 - q^8 + 8q^9 - 8q^{10} - 2q^{11} \\ &\quad +14q^{12} - 14q^{13} - 4q^{14} + 24q^{15} - 23q^{16} - 6q^{17} + 40q^{18} - 38q^{19} \\ &\quad -10q^{20} + 63q^{21} - 60q^{22} - 16q^{23} + 98q^{24} - 92q^{25} - 24q^{26} + 150q^{27} \\ &\quad -140q^{128} - 36q^{29} + 224q^{30} + \ldots \quad . \end{aligned}$$

The following table verifies the case $n = 4$ in the Theorem 2.2.

$P_7(4)$=14 $=a_{12}$	$P_5(4)$ =+14 $=-a_{13}$	$p_1(3)=4$ $=-a_{14}$
$4{=}3_r+1_r{=}3_r+1_g{=}3_g+1_g{=}$ $3_g+1_r{=}2{+}2{=}2+1_r+1_r{=}$ $2+1_r+1_g{=}2+1_g+1_g{=}$ $1_r+1_r+1_r+1_r{=}1_r+1_r+1_r+1_g{=}$ $1_r+1_r+1_g+1_g{=}1_r+1_g+1_g$ $+1_g=1_g+1_g+1_g+1_g$	$4{=}3_r+1_r{=}3_r+1_g{=}3_g+1_g{=}$ $3_g+1_r{=}2{+}2{=}2+1_r+1_r{=}$ $2+1_r+1_g{=}2+1_g+1_g{=}$ $1_r+1_r+1_r+1_r{=}1_r+1_r+1_r$ $+1_g=1_r+1_r+1_g+1_g{=}1_r+1_g$ $+1_g+1_g{=}1_g+1_g+1_g+1_g$.	$3_r{=}3_g=2+1$ =1+1+1

Corollary 2.1 *With a_n given by (1.8), $a_2=0$. The remaining a_n satisfy, for $n\geq 0$,*

$$a_{3n}>0,\ a_{3n+1}<0,\ a_{3n+2}\leq 0.$$

Proof This follows directly from Theorem 2.2. □

Lemma 2.2 *We have*

$$M(q)=\frac{1}{\psi(q^3)}\left[\sum_{n=-\infty}^{\infty}q^{6n^2-n}+\sum_{n=-\infty}^{\infty}q^{6n^2-5n+1}\right]. \tag{2.10}$$

Proof We have [6, p. 345]

$$\frac{1}{G(q)}=\frac{1}{\psi(q^3)}\left[\psi(q^{1/3})-q^{1/3}\psi(q^3)\right]$$

Upon using (1.4) in the above, we obtain

$$M(q)=\frac{1}{\psi(q^3)}\left[\sum_{n=-\infty}^{\infty}q^{\frac{2n^2-n}{3}}-\sum_{n=-\infty}^{\infty}q^{\frac{18n^2-9n+1}{3}}\right].$$

Converting the first series in the right hand side of the above into sum of three series and replacing n by $-n$ in the second series, we obtain

$$M(q)=\frac{1}{\psi(q^3)}\left[\sum_{n=-\infty}^{\infty}q^{6n^2-n}+\sum_{n=-\infty}^{\infty}q^{6n^2-5n+1}\right].$$ □

Theorem 2.3 *Let b_n be as defined in (1.7). Then*

$$\sum_{n=0}^{\infty}b_{3n}q^n=\frac{(q;q)_\infty}{(q^2;q^2)_\infty^2}\left[\sum_{n=-\infty}^{\infty}q^{18n^2+n}+q^4\sum_{n=-\infty}^{\infty}q^{18n^2+17n}\right], \tag{2.11}$$

$$\sum_{n=0}^{\infty}b_{3n+1}q^n=\frac{(q;q)_\infty}{(q^2;q^2)_\infty^2}\left[\sum_{n=-\infty}^{\infty}q^{18n^2+5n}+q^2\sum_{n=-\infty}^{\infty}q^{18n^2+13n}\right] \tag{2.12}$$

and

$$\sum_{n=0}^{\infty} b_{3n+2}q^n = \frac{(q;q)_\infty}{(q^2;q^2)_\infty^2}\left[\sum_{n=-\infty}^{\infty} q^{18n^2+7n} + q\sum_{n=-\infty}^{\infty} q^{18n^2+11n}\right]. \tag{2.13}$$

Proof Following similar steps used in the proof of Theorem 2.1, for (2.10) we have for $0 \le k \le 2$,

$$\sum_{n=0}^{\infty} b_{3n+k}q^n = \frac{1}{3}\frac{1}{\psi(q)}\sum_{j=0}^{2}\left[\sum_{n=-\infty}^{\infty} t^{(6n^2-n-k)j}q^{\frac{6n^2-n-k}{3}} + \sum_{n=-\infty}^{\infty} t^{(6n^2-5n+1-k)j}q^{\frac{6n^2-5n+1-k}{3}}\right].$$

Now $6n^2 - n - k \equiv 0 \pmod 3$ for $n \equiv -k \pmod 3$. And for the second summation $6n^2 - 5n + 1 - k \equiv 0 \pmod 3$ for $2n \equiv 1 - k \pmod 3$. It therefore follows from the above that

$$\begin{aligned}\sum_{n=0}^{\infty} b_{3n+k}q^n &= \frac{1}{\psi(q)}\left[\sum_{n=-\infty}^{\infty} q^{\frac{6(3n-k)^2-(3n-k)-k}{3}} + \sum_{n=-\infty}^{\infty} q^{\frac{6(6n-4k+4)^2-10(6n-4k+4)+4-4k}{12}}\right]\\ &= \frac{1}{\psi(q)}\left[\sum_{n=-\infty}^{\infty} q^{18n^2-12nk-n+2k^2} + \sum_{n=-\infty}^{\infty} q^{18n^2-24nk+19n+8k^2-13k+5}\right]. \end{aligned}\tag{2.14}$$

□

The identities (2.11)- (2.13) now follow by setting $k = 0, 1, 2$ respectively in (2.14). In the case (2.11) the index of the summation needs to be changed by replacing n to $-n$ in the first summation and in the second summation by n to $n+1$ and then by replacing n to $-n$. For the case (2.12), the index of the summation in the second summation is to be changed by replacing n to $-n$. In the case (2.13) the index of the summation in the first and second series are to be changed by replacing n to $n+1$.

Theorem 2.4 *Let $P_s(n)$ denote the number of partitions of n into part such that odd parts are not congruent to $\pm s$ (mod 36) and the even part congruent to $\pm 4, \pm 8, \pm 12, \pm 16$ (mod 36). Then*

$$b_{3n} = (-1)^n\left[P_{17}(n) + P_1(n-4)\right], \tag{2.15}$$

$$b_{3n+1} = (-1)^n\left[P_{13}(n) + P_5(n-2)\right] \tag{2.16}$$

and

$$b_{3n+2} = (-1)^n\left[P_{11}(n) - P_7(n-1)\right]. \tag{2.17}$$

Proof It is easy to see that

$$\frac{(-q;-q)_\infty}{(q^2;q^2)_\infty^2} = \frac{1}{(q;q^2)_\infty(q^4;q^4)_\infty}, \tag{2.18}$$

Replacing q to q^2 in (2.9) and then setting $z = -q$, $z = -q^{17}$, $-q^5$, $-q^{13}$, $-q^7$, $-q^{11}$ respectively, we obtain

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{18n^2+n} = (q^{17\pm}, q^{36}; q^{36})_\infty,$$

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{18n^2+17n}=(q^{1\pm},q^{36};q^{36})_\infty,$$

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{18n^2+5n}=(q^{13\pm},q^{36};q^{36})_\infty,$$

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{18n^2+13n}=(q^{5\pm},q^{36};q^{36})_\infty,$$

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{18n^2+7n}=(q^{11\pm},q^{36};q^{36})_\infty$$

and

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{18n^2+11n}=(q^{7\pm},q^{36};q^{36})_\infty.$$

Now changing q to $-q$ in (2.11), (2.12) and (2.13) and then using (2.18) and the above, we deduce that

$$\sum_{n=0}^{\infty}(-1)^n b_{3n}q^n=\frac{(q^{17\pm},q^{36};q^{36})_\infty+q^4(q^{1\pm},q^{36};q^{36})_\infty}{(q;q^2)_\infty(q^4;q^4)_\infty}, \tag{2.19}$$

$$\sum_{n=0}^{\infty}(-1)^n b_{3n+1}q^n=\frac{(q^{13\pm},q^{36};q^{36})_\infty+q^2(q^{5\pm},q^{36};q^{36})_\infty}{(q;q^2)_\infty(q^4;q^4)_\infty} \tag{2.20}$$

and

$$\sum_{n=0}^{\infty}(-1)^n b_{3n+2}q^n=\frac{(q^{11\pm},q^{36};q^{36})_\infty-q(q^{7\pm},q^{36};q^{36})_\infty}{(q;q^2)_\infty(q^4;q^4)_\infty}. \tag{2.21}$$

Now (2.15), (2.16) and (2.17) follow from (2.19), (2.20) and (2.21) respectively. □

Example 2.2 By using Maple we have been able to find the following series expansion for $M(q)$:

$$\begin{aligned} M(q) \quad = \quad & 1+q+q^2-q^3-q^4+q^6+2q^7-2q^9-3q^{10}-q^{11}+4q^{12}+4q^{13}+q^{14}\\ & -4q^{15}-6q^{16}-q^{17}+5q^{18}+8q^{19}+q^{20}-8q^{21}-10q^{22}-2q^{23}+11q^{24}\\ & +14q^{25}+4q^{26}-14q^{27}-19q^{28}-4q^{29}+17q^{30}\cdots. \end{aligned}$$

The following table verifies the case $n=5$ in Theorem 2.4:

$(-1)^5[P_{17}(5)+P_1(1)]$ $=-4=a_{15}$		$(-1)^5[P_{13}(5)+P_5(3)]$ $=-6=a_{16}$		$(-1)^5[P_{11}(5)-P_7(4)]$ $=-1=a_{17}$	
$P_{17}(5)$	$P_1(1)$	$P_{13}(5)$	$P_5(3)$	$P_{11}(5)$	$P_7(4)$
5		5	3	5	4
=4+1		=4+1	=1+1+1	=4+1	=3+1
=3+1+1		=3+1+1		=3+1+1	=1+1+1+1
=1+1+1+1+1		1+1+1+1+1		=1+1+1+1+1	

Corollary 2.2([7]) *With b_n given by (1.7), $b_5 = 0$ and $b_8 = 0$. The remaining b_n satisfy for $n \geq 0$,*

$$b_{6n} > 0, \qquad b_{6n+1} > 0, \qquad b_{6n+2} > 0$$
$$b_{6n+3} < 0, \qquad b_{6n+4} < 0, \qquad b_{6n+5} < 0.$$

Proof The result clearly follows from Theorem 2.4. □

§3. Further Identities of $G(q)$ and $M(q)$

Following the notation in [17], we define

$$\begin{aligned} L(q) : &= \sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q;q^2)_n}{(q^4;q^4)_n} \\ &= \frac{f(-q,-q^5)}{\psi(-q)} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} N(q) &:= \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^4;q^4)_n} \\ &= \frac{f(-q^3,-q^3)}{\psi(-q)}. \end{aligned} \tag{3.2}$$

The two identities on the right of (3.1) and (3.2) are the cubic identities due to G. E. Andrews [2] and L. J. Slater [16] respectively. Andrews [2] shown that

$$q^{1/3}G(q) = q^{1/3}\frac{L(q)}{N(q)}. \tag{3.3}$$

We note that

$$L(q) = \frac{f_6^2}{f_3 f_4}, \qquad N(q) = \frac{f_3^2 f_2}{f_1 f_4 f_6}$$

and

$$f(q) = \frac{f_2^3}{f_1 f_4} \tag{3.4}$$

where $f_n := (q^n;q^n)_\infty$.

Lemma 3.1([17]) *We have*

$$L(-q)N(q) - L(q)N(-q) = 2q\frac{f_2 f_{12}^4}{f_4^3 f_6^2} \tag{3.5}$$

and

$$L(-q)N(q) + L(q)N(-q) = 2\frac{f_4}{f_2}. \tag{3.6}$$

Theorem 3.1 *We have*

$$\sum_{n=0}^{\infty} a_{2n} q^n = \frac{(q^2, q^4, q^6; q^6)_\infty^2}{(q^3, q^3, q^6; q^6)_\infty^2} = \frac{\left[\sum\limits_{n=-\infty}^{\infty} (-1)^n q^{3n^2-n}\right]^2}{\left[\sum\limits_{n=-\infty}^{\infty} (-1)^n q^{3n^2}\right]^2}, \tag{3.7}$$

$$\sum_{n=0}^{\infty} a_{2n+1} q^n = \frac{(q, q^5, q^6; q^6)_\infty^2}{(q^3, q^3, q^6; q^6)_\infty^2} = \frac{\left[\sum\limits_{n=-\infty}^{\infty} (-1)^n q^{3n^2-2n}\right]^2}{\left[\sum\limits_{n=-\infty}^{\infty} (-1)^n q^{3n^2}\right]^2}, \tag{3.8}$$

$$\sum_{n=0}^{\infty} b_{2n} q^n = \frac{(q^2, q^4, q^6; q^6)_\infty^2}{(q, q^3, q^3, q^5, q^6, q^6; q^6)_\infty} = \frac{\left[\sum\limits_{n=-\infty}^{\infty} (-1)^n q^{3n^2-n}\right]^2}{\left[\sum\limits_{n=-\infty}^{\infty} (-1)^n q^{3n^2-2n}\right]\left[\sum\limits_{n=-\infty}^{\infty} (-1)^n q^{3n^2}\right]} \tag{3.9}$$

and

$$\sum_{n=0}^{\infty} b_{2n+1} q^n = \frac{(q, q^5, q^6; q^6)_\infty}{(q^3, q^3, q^6; q^6)_\infty} = \frac{\sum\limits_{n=-\infty}^{\infty} (-1)^n q^{3n^2-2n}}{\sum\limits_{n=-\infty}^{\infty} (-1)^n q^{3n^2}}. \tag{3.10}$$

Proof We have

$$\begin{aligned}\sum_{n=0}^{\infty} a_{2n} q^{2n} &= \frac{1}{2}\left[G(q) + G(-q)\right] = \frac{1}{2}\left[\frac{L(q)}{N(q)} + \frac{L(-q)}{M(-q)}\right] \\ &= \frac{1}{2}\frac{L(q)M(-q) + L(-q)N(q)}{N(q)M(-q)},\end{aligned}$$

which on employing (3.6) and (3.4) yields

$$\sum_{n=0}^{\infty} a_{2n} q^{2n} = \frac{(q^4, q^8, q^{12}; q^{12})_\infty^2}{(q^6, q^6, q^{12}; q^{12})_\infty^2}.$$

Changing q to $q^{1/2}$ in the above, we obtain the first equality of (3.7). The second equality follows by appealing to (1.1). Similarly, we have

$$\begin{aligned}\sum_{n=0}^{\infty} a_{2n+1} q^{2n} &= \frac{1}{2q}\left[G(q) - G(-q)\right] = \frac{1}{2q}\left[\frac{L(q)}{N(q)} - \frac{L(-q)}{M(-q)}\right] \\ &= \frac{1}{2q}\frac{L(q)M(-q) - L(-q)N(q)}{N(q)M(-q)}.\end{aligned}$$

Now employing (3.5), (3.4) and (1.1) gives

$$\sum_{n=0}^{\infty} a_{2n+1} q^{2n} = -\frac{(q^2, q^{10}, q^{12}; q^{12})_\infty^2}{(q^6, q^6, q^{12}; q^{12})_\infty^2}.$$

The first equality in result (3.8) now follows by changing q to $q^{1/2}$ in the above and the second equality follows by employing (1.1). By similar arguments, one can derive (3.9) and (3.10). □

Theorem 3.2 *For* $|q| < 1$

$$\frac{\sum_{n=0}^{\infty} a_{2n} q^n}{\sum_{n=0}^{\infty} a_{2n+1} q^n} = -\frac{\sum_{n=0}^{\infty} b_{2n} q^n}{\sum_{n=0}^{\infty} b_{2n+1} q^n}.$$

Proof Follows from Theorem 3.1. □

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Some Families of Chromatically Unique 5-Partite Graphs

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Abstract: Let $P(G, \lambda)$ be the chromatic polynomial of a graph G. Two graphs G and H are said to be chromatically equivalent, denoted $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. We write $[G] = \{H | H \sim G\}$. If $[G] = \{G\}$, then G is said to be chromatically unique. In this paper, we first characterize certain complete 5-partite graphs G with $5n + i$ vertices for $i = 1, 2, 3$ according to the number of 6-independent partitions of G. Using these results, we investigate the chromaticity of G with certain star or matching deleted. As a by-product, many new families of chromatically unique complete 5-partite graphs G with certain star or matching deleted are obtained.

Key Words: Chromatic polynomial, chromatically closed, chromatic uniqueness.

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§1. Introduction

All graphs considered here are simple and finite. For a graph G, let $P(G, \lambda)$ be the chromatic polynomial of G. Two graphs G and H are said to be *chromatically equivalent* (or simply $\chi-$equivalent), symbolically $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. The equivalence class determined by G under $\sim$ is denoted by $[G]$. A graph G is *chromatically unique* (or simply $\chi-$unique) if $H \cong G$ whenever $H \sim G$, i.e, $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be $\chi-$*closed*. Many families of χ-unique graphs are known (see [3,4]).

For a graph G, let $V(G)$, $E(G)$, $t(G)$ and $\chi(G)$ be the vertex set, edge set, number of triangles and chromatic number of G, respectively. Let O_n be an edgeless graph with n vertices.

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Let $Q(G)$ and $K(G)$ be the number of induced subgraph C_4 and complete subgraph K_4 in G. Let S be a set of s edges in G. By $G-S$ (or $G-s$) we denote the graph obtained from G by deleting all edges in S, and $\langle S\rangle$ the graph induced by S. For $t \geqslant 2$ and $1 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_t$, let $K(n_1, n_2, \cdots, n_t)$ be a complete $t-$partite graph with partition sets V_i such that $|V_i| = n_i$ for $i = 1, 2, \cdots, t$. In [2,5-7,9-11,13-15], the authors proved that certain families of complete t-partite graphs ($t = 2, 3, 4, 5$) with a matching or a star deleted are χ-unique. In particular, Zhao et al. [13,14] investigated the chromaticity of complete 5-partite graphs G of $5n$ and $5n+4$ vertices with certain star or matching deleted. As a continuation, in this paper, we characterize certain complete 5-partite graphs G with $5n+i$ vertices for $i = 1, 2, 3$ according to the number of 6-independent partitions of G. Using these results, we investigate the chromaticity of G with certain star or matching deleted. As a by-product, many new families of chromatically unique complete 5-partite graphs with certain star or matching deleted are obtained.

§2. Some Lemmas and Notations

Let $\mathcal{K}^{-s}(n_1, n_2, \cdots, n_t)$ be the family $\{K(n_1, n_2, \cdots, n_t) - S|\ S \subset E(K(n_1, n_2, \cdots, n_t))$ and $|S| = s\}$. For $n_1 \geqslant s+1$, we denote by $K_{i,j}^{-K_{1,s}}(n_1, n_2, \cdots, n_t)$ (respectively, $K_{i,j}^{-sK_2}(n_1, n_2, \cdots, n_t)$) the graph in $K^{-s}(n_1, n_2, \cdots, n_t)$ where the s edges in S induced a $K_{1,s}$ with center in V_i and all the end vertices in V_j (respectively, a matching with end vertices in V_i and V_j).

For a graph G and a positive integer r, a partition $\{A_1, A_2, \cdots, A_r\}$ of $V(G)$, where r is a positive integer, is called an *r-independent partition* of G if every A_i is independent of G. Let $\alpha(G, r)$ denote the number of r-independent partitions of G. Then, we have $P(G, \lambda) = \sum_{r=1}^{p} \alpha(G, r)(\lambda)_r$, where $(\lambda)_r = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-r+1)$ (see [8]). Therefore, $\alpha(G, r) = \alpha(H, r)$ for each $r = 1, 2, \cdots$, if $G \sim H$.

For a graph G with p vertices, the polynomial $\sigma(G, x) = \sum_{r=1}^{p} \alpha(G, r)x^r$ is called the σ-polynomial of G (see [1]). Clearly, $P(G, \lambda) = P(H, \lambda)$ implies that $\sigma(G, x) = \sigma(H, x)$ for any graphs G and H.

For disjoint graphs G and H, $G + H$ denotes the disjoint union of G and H. The join of G and H denoted by $G \vee H$ is defined as follows: $V(G \vee H) = V(G) \cup V(H)$; $E(G \vee H) = E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$. For notations and terminology not defined here, we refer to [12].

Lemma 2.1 (Koh and Teo [3]) *Let G and H be two graphs with $H \sim G$, then $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$, $t(G) = t(H)$ and $\chi(G) = \chi(H)$. Moreover, $\alpha(G, r) = \alpha(H, r)$ for $r = 1, 2, 3, 4, \cdots$, and $2K(G) - Q(G) = 2K(H) - Q(H)$. Note that $\chi(G) = 3$ then $G \sim H$ implies that $Q(G) = Q(H)$.*

Lemma 2.2(Brenti [1]) *Let G and H be two disjoint graphs. Then*

$$\sigma(G \vee H, x) = \sigma(G, x)\sigma(H, x).$$

In particular,

$$\sigma(K(n_1, n_2, \cdots, n_t), x) = \prod_{i=1}^{t} \sigma(O_{n_i}, x)$$

Lemma 2.3(Zhao [13]) *Let $G = K(n_1, n_2, n_3, n_4, n_5)$ and S be a set of some s edges of G. If $H \sim G - S$, then there is a complete graph $F = K(p_1, p_2, p_3, p_4, p_5)$ and a subset S' of $E(F)$ of some s' of F such that $H = F - S'$ with $|S'| = s' = e(F) - e(G) + s$.*

Let $x_1 \leqslant x_2 \leqslant x_3 \leqslant x_4 \leqslant x_5$ be positive integers, $\{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}\} = \{x_1, x_2, x_3, x_4, x_5\}$. If there exists two elements x_{i_1} and x_{i_2} in $\{x_1, x_2, x_3, x_4, x_5\}$ such that $x_{i_2} - x_{i_1} \geqslant 2$, $H' = K(x_{i_1} + 1, x_{i_2} - 1, x_{i_3}, x_{i_4}, x_{i_5})$ is called an *improvement* of $H = K(x_1, x_2, x_3, x_4, x_5)$.

Lemma 2.4 (Zhao et al. [13]) *Suppose $x_1 \leqslant x_2 \leqslant x_3 \leqslant x_4 \leqslant x_5$ and $H' = K(x_{i_1} + 1, x_{i_2} - 1, x_{i_3}, x_{i_4}, x_{i_5}\}$ is an improvement of $H = K(x_1, x_2, x_3, x_4, x_5)$, then*

$$\alpha(H, 6) - \alpha(H', 6) = 2^{x_{i_2}-2} - 2^{x_{i_1}-1} \geqslant 2^{x_{i_1}-1}.$$

Let $G = K(n_1, n_2, n_3, n_4, n_5)$. For a graph $H = G - S$, where S is a set of some s edges of G, define $\alpha'(H) = \alpha(H, 6) - \alpha(G, 6)$. Clearly, $\alpha'(H) \geqslant 0$.

Lemma 2.5 (Zhao et al. [13]) *Let $G = K(n_1, n_2, n_3, n_4, n_5)$. Suppose that min $\{n_i | i = 1, 2, 3, 4, 5\} \geqslant s + 1 \geqslant 1$ and $H = G - S$, where S is a set of some s edges of G, then*

$$s \leqslant \alpha'(H) = \alpha(H, 6) - \alpha(G, 6) \leqslant 2^s - 1,$$

and $\alpha'(H) = s$ iff the set of end-vertices of any $r \geqslant 2$ edges in S is not independent in H, and $\alpha'(H) = 2^s - 1$ iff S induces a star $K_{1,s}$ and all vertices of $K_{1,s}$ other than its center belong to a same A_i.

Lemma 2.6(Dong et al. [2]) *Let n_1, n_2 and s be positive integers with $3 \leqslant n_1 \leqslant n_2$, then*

(1) $K_{1,2}^{-K_{1,s}}(n_1, n_2)$ is χ-unique for $1 \leqslant s \leqslant n_2 - 2$,

(2) $K_{2,1}^{-K_{1,s}}(n_1, n_2)$ is χ-unique for $1 \leqslant s \leqslant n_1 - 2$, and

(3) $K^{-sK_2}(n_1, n_2)$ is χ-unique for $1 \leqslant s \leqslant n_1 - 1$.

For a graph $G \in K^{-s}(n_1, n_2, \cdots, n_t)$, we say an induced C_4 subgraph of G is of Type 1 (respectively Type 2 and Type 3) if the vertices of the induced C_4 are in exactly two (respectively three and four) partite sets of $V(G)$. An example of induced C_4 of Types 1, 2 and 3 are shown in Figure 1.

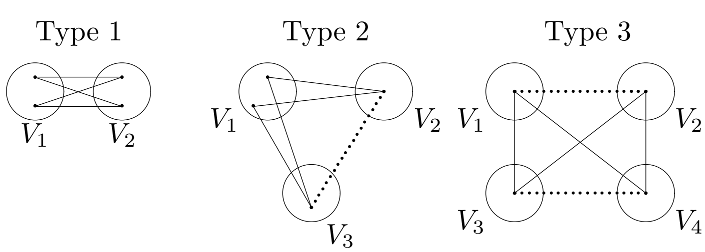

FIGURE 1. Three types of induced C_4

Suppose G is a graph in $K^{-s}(n_1, n_2, \cdots, n_t)$. Let S_{ij} $(1 \leqslant i \leqslant t, 1 \leqslant j \leqslant t)$ be a subset of S such that each edge in S_{ij} has an end-vertex in V_i and another end-vertex in V_j with $|S_{ij}| = s_{ij} \geqslant 0$.

Lemma 2.7 (Lau and Peng [6]) *For integer $t \geqslant 3$, Let $F = K(n_1, n_2, \cdots, n_t)$ be a complete t-partite graph and let $G = F - S$ where S is a set of s edges in F. If S induces a matching in F, then*

$$\begin{aligned} Q(G) &= Q(F) - \sum_{1\leqslant i<j\leqslant t} (n_i-1)(n_j-1)s_{ij} + \binom{s}{2} - \sum_{1\leqslant i<j<l\leqslant t} s_{ij}s_{il} - \\ &\quad \sum_{\substack{1\leqslant i<j\leqslant t\\ 1\leqslant k<l\leqslant t\\ i<k}} s_{ij}s_{kl} + \sum_{1\leqslant i<j\leqslant t}\left[s_{ij}\sum_{k\notin\{i,j\}}\binom{n_k}{2}\right] + \sum_{\substack{1\leqslant i<j\leqslant t\\ 1\leqslant i<k<l\leqslant t\\ j\notin\{k,l\}}} s_{ij}s_{kl}, \end{aligned}$$

and

$$K(G) = K(F) - \sum_{1\leqslant i<j\leqslant t}\left[s_{ij}\sum_{\substack{1\leqslant k<l\leqslant t\\ \{i,j\}\cap\{k,l\}=\emptyset}} n_k n_l\right] + \sum_{\substack{1\leqslant i<j\leqslant t\\ 1\leqslant i<k<l\leqslant t\\ j\notin\{k,l\}}} s_{ij}s_{kl}.$$

By using Lemma 2.7, we obtain the following.

Lemma 2.8 *Let $F = K(n_1, n_2, n_3, n_4, n_5)$ be a complete 5-partite graph and let $G = F - S$ where S is a set of s edges in F. If S induces a matching in F, then*

$$\begin{aligned} Q(G) &= Q(F) - \sum_{1\leqslant i<j\leqslant 5}(n_i-1)(n_j-1)s_{ij} + \binom{s}{2} - s_{12}(s_{13}+s_{14}+s_{15}+s_{23} \\ &\quad + s_{24}+s_{25}) - s_{13}(s_{14}+s_{15}+s_{23}+s_{34}+s_{35}) - s_{14}(s_{15}+s_{24}+s_{34}+s_{45}) \\ &\quad - s_{15}(s_{25}+s_{35}+s_{45}) - s_{23}(s_{24}+s_{25}+s_{34}+s_{35}) - s_{24}(s_{25}+s_{34}+s_{45}) \\ &\quad - s_{25}(s_{35}+s_{45}) - s_{34}(s_{35}+s_{45}) - s_{35}s_{45} + \sum_{1\leqslant i<j\leqslant 5}\left[s_{ij}\sum_{k\notin\{i,j\}}\binom{n_k}{2}\right], \\ K(G) &= K(F) - \sum_{1\leqslant i<j\leqslant 5}\left[s_{ij}\sum_{\substack{1\leqslant k<l\leqslant 5\\ \{i,j\}\cap\{k,l\}=\emptyset}} n_k n_l\right] + s_{12}(s_{34}+s_{35}+s_{45}) \\ &\quad + s_{13}(s_{24}+s_{25}+s_{45}) + s_{14}(s_{23}+s_{25}+s_{35}) + s_{15}(s_{23}+s_{24}+s_{34}) + s_{23}s_{45} \\ &\quad + s_{24}s_{35} + s_{25}s_{34}. \end{aligned}$$

Moreover, these equalities hold if and only if each edge in S joins vertices in the same two partite sets of smallest size in F.

§3. Characterization

In this section, we shall characterize certain complete 5-partite graph $G = K(n_1, n_2, n_3, n_4, n_5)$ according to the number of 6-independent partitions of G where $n_5 - n_1 \leqslant 4$.

Theorem 3.1 *Let $G = K(n_1, n_2, n_3, n_4, n_5)$ be a complete 5-partite graph such that $n_1 + n_2 + n_3 + n_4 + n_5 = 5n + 1$ and $n_5 - n_1 \leqslant 4$. Define $\theta(G) = [\alpha(G, 6) - 2^{n+1} - 2^n + 5]/2^{n-2}$. Then*

(i) $\theta(G) = 0$ *if and only if* $G = K(n, n, n, n, n + 1)$;

(ii) $\theta(G) = 1$ *if and only if* $G = K(n - 1, n, n, n + 1, n + 1)$;

(iii) $\theta(G) = 2$ *if and only if* $G = K(n - 1, n - 1, n + 1, n + 1, n + 1)$;

(iv) $\theta(G) = 2\frac{1}{2}$ *if and only if* $G = K(n - 2, n, n + 1, n + 1, n + 1)$;

(v) $\theta(G) = 3$ *if and only if* $G = K(n - 1, n, n, n, n + 2)$;

(vi) $\theta(G) = 4$ *if and only if* $G = K(n - 1, n - 1, n, n + 1, n + 2)$;

(vii) $\theta(G) = 4\frac{1}{4}$ *if and only if* $G = K(n - 3, n + 1, n + 1, n + 1, n + 1)$;

(viii) $\theta(G) = 4\frac{1}{2}$ *if and only if* $G = K(n - 2, n, n, n + 1, n + 2)$;

(ix) $\theta(G) = 5\frac{1}{2}$ *if and only if* $G = K(n - 2, n - 1, n + 1, n + 1, n + 2)$;

(x) $\theta(G) = 7$ *if and only if* $G = K(n - 1, n - 1, n - 1, n + 2, n + 2)$;

(xi) $\theta(G) = 7\frac{1}{2}$ *if and only if* $G = K(n - 2, n - 1, n, n + 2, n + 2)$;

(xii) $\theta(G) = 9$ *if and only if* $G = K(n - 2, n - 2, n + 1, n + 2, n + 2)$;

(xiii) $\theta(G) = 10$ *if and only if* $G = K(n - 1, n - 1, n, n, n + 3)$;

(xiv) $\theta(G) = 11$ *if and only if* $G = K(n - 1, n - 1, n - 1, n + 1, n + 3)$.

Proof In order to complete the proof of the theorem, we first give a table for the θ-value of various complete 5-partite graphs with $5n + 1$ vertices as shown in Table 1.

(i) G_1 is the improvement of G_2 and G_3 with $\theta(G_2) = 1$ and $\theta(G_3) = 3$;

(ii) G_2 is the improvement of G_3, G_4, G_5, G_6 and G_7 with $\theta(G_3) = 3$, $\theta(G_4) = 2$, $\theta(G_5) = 4$, $\theta(G_6) = 2\frac{1}{2}$ and $\theta(G_7) = 4\frac{1}{2}$;

(iii) G_3 is the improvement of G_5, G_7, G_8 and G_9 with $\theta(G_5) = 4$, $\theta(G_7) = 4\frac{1}{2}$ and $\theta(G_8) = 10$ and $\theta(G_9) = 10\frac{1}{2}$;

(iv) G_4 is the improvement of G_5, G_6 and G_{10} with $\theta(G_5) = 4$, $\theta(G_6) = 2\frac{1}{2}$ and $\theta(G_{10}) = 5\frac{1}{2}$;

(v) G_5 is the improvement of G_7, G_8, G_{10}, G_{11}, G_{12}, G_{13} and G_{14} with $\theta(G_7) = 4\frac{1}{2}$, $\theta(G_8) = 10$, $\theta(G_{10}) = 5\frac{1}{2}$, $\theta(G_{11}) = 7$, $\theta(G_{12}) = 11$, $\theta(G_{13}) = 7\frac{1}{2}$ and $\theta(G_{14}) = 11\frac{1}{2}$;

(vi) G_6 is the improvement of G_7, G_{10}, G_{15} and G_{16} with $\theta(G_7) = 4\frac{1}{2}$, $\theta(G_{10}) = 5\frac{1}{2}$, $\theta(G_{15}) = 4\frac{1}{4}$ and $\theta(G_{16}) = 6\frac{1}{4}$;

G_i $(1 \leqslant i \leqslant 21)$	$\theta(G_i)$	G_i $(22 \leqslant i \leqslant 41)$	$\theta(G_i)$
$G_1 = K(n,n,n,n,n+1)$	0	$G_{22} = K(n-2,n-2,n+1,n+2,n+2)$	9
$G_2 = K(n-1,n,n,n+1,n+1)$	1	$G_{23} = K(n-2,n-2,n+1,n+1,n+3)$	13
$G_3 = K(n-1,n,n,n,n+2)$	3	$G_{24} = K(n-3,n-1,n+1,n+2,n+2)$	$9\frac{1}{4}$
$G_4 = K(n-1,n-1,n+1,n+1,n+1)$	2	$G_{25} = K(n-3,n-1,n+1,n+1,n+3)$	$13\frac{1}{4}$
$G_5 = K(n-1,n-1,n,n+1,n+2)$	4	$G_{26} = K(n-2,n-1,n-1,n+2,n+3)$	$14\frac{1}{2}$
$G_6 = K(n-2,n,n+1,n+1,n+1)$	$2\frac{1}{2}$	$G_{27} = K(n-2,n-1,n-1,n+1,n+4)$	$26\frac{1}{2}$
$G_7 = K(n-2,n,n,n+1,n+2)$	$4\frac{1}{2}$	$G_{28} = K(n-2,n-2,n,n+2,n+3)$	15
$G_8 = K(n-1,n-1,n,n,n+3)$	10	$G_{29} = K(n-3,n-1,n,n+2,n+3)$	$15\frac{1}{4}$
$G_9 = K(n-2,n,n,n,n+3)$	$10\frac{1}{2}$	$G_{30} = K(n-4,n+1,n+1,n+1,n+2)$	$8\frac{1}{8}$
$G_{10} = K(n-2,n-1,n+1,n+1,n+2)$	$5\frac{1}{2}$	$G_{31} = K(n-4,n,n+1,n+2,n+2)$	$10\frac{1}{8}$
$G_{11} = K(n-1,n-1,n-1,n+2,n+2)$	7	$G_{32} = K(n-4,n,n+1,n+1,n+3)$	$14\frac{1}{8}$
$G_{12} = K(n-1,n-1,n-1,n+1,n+3)$	11	$G_{33} = K(n-4,n,n,n+2,n+3)$	$16\frac{1}{8}$
$G_{13} = K(n-2,n-1,n,n+2,n+2)$	$7\frac{1}{2}$	$G_{34} = K(n-3,n-2,n+2,n+2,n+2)$	$12\frac{3}{4}$
$G_{14} = K(n-2,n-1,n,n+1,n+3)$	$11\frac{1}{2}$	$G_{35} = K(n-3,n-2,n+1,n+2,n+3)$	$16\frac{3}{4}$
$G_{15} = K(n-3,n+1,n+1,n+1,n+1)$	$4\frac{1}{4}$	$G_{36} = K(n-4,n-1,n+2,n+2,n+2)$	$13\frac{1}{8}$
$G_{16} = K(n-3,n,n+1,n+1,n+2)$	$6\frac{1}{4}$	$G_{37} = K(n-4,n-1,n+1,n+2,n+3)$	$17\frac{1}{8}$
$G_{17} = K(n-3,n,n,n+2,n+2)$	$8\frac{1}{4}$	$G_{38} = K(n-5,n+1,n+1,n+2,n+2)$	$12\frac{1}{16}$
$G_{18} = K(n-3,n,n,n+1,n+3)$	$12\frac{1}{4}$	$G_{39} = K(n-5,n+1,n+1,n+1,n+3)$	$16\frac{1}{16}$
$G_{19} = K(n-1,n-1,n-1,n,n+4)$	25	$G_{40} = K(n-5,n,n+2,n+2,n+2)$	$14\frac{1}{16}$
$G_{20} = K(n-2,n-1,n,n,n+4)$	$25\frac{1}{2}$	$G_{41} = K(n-5,n,n+1,n+2,n+3)$	$18\frac{1}{16}$
$G_{21} = K(n-3,n,n,n,n+4)$	$26\frac{1}{4}$		

Table 1 Complete 5-partite graphs with $5n+1$ vertices.

By the definition of improvement, we have the followings:

(vii) G_7 is the improvement of G_9, G_{10}, G_{13}, G_{14}, G_{16}, G_{17} and G_{18} with $\theta(G_9) = 10\frac{1}{2}$, $\theta(G_{10}) = 5\frac{1}{2}$, $\theta(G_{13}) = 7\frac{1}{2}$, $\theta(G_{14}) = 11\frac{1}{2}$, $\theta(G_{16}) = 6\frac{1}{4}$, $\theta(G_{17}) = 8\frac{1}{4}$and $\theta(G_{18}) = 12\frac{1}{4}$;

(viii) G_8 is the improvement of G_9, G_{12}, G_{14}, G_{19} and G_{20} with $\theta(G_9) = 10\frac{1}{2}$, $\theta(G_{12}) = 11$, $\theta(G_{14}) = 11\frac{1}{2}$, $\theta(G_{19}) = 25$ and $\theta(G_{20}) = 25\frac{1}{2}$;

(ix) G_9 is the improvement of G_{14}, G_{18}, G_{20} and G_{21} with $\theta(G_{14}) = 11\frac{1}{2}$, $\theta(G_{18}) = 12\frac{1}{4}$, $\theta(G_{20}) = 25\frac{1}{2}$ and $\theta(G_{21}) = 26\frac{1}{4}$;

(x) G_{10} is the improvement of G_{13}, G_{14}, G_{16}, G_{22}, G_{23}, G_{24} and G_{25} with $\theta(G_{13}) = 7\frac{1}{2}$, $\theta(G_{14}) = 11\frac{1}{2}$, $\theta(G_{16}) = 6\frac{1}{4}$, $\theta(G_{22}) = 9$, $\theta(G_{23}) = 13$, $\theta(G_{24}) = 9\frac{1}{4}$ and $\theta(G_{25}) = 13\frac{1}{4}$;

(xi) G_{11} is the improvement of G_{12}, G_{13} and G_{26} with $\theta(G_{12}) = 11$, $\theta(G_{13}) = 7\frac{1}{2}$ and $\theta(G_{26}) = 14\frac{1}{2}$;

(xii) G_{12} is the improvement of G_{14}, G_{19}, G_{26} and G_{27} with $\theta(G_{14}) = 11\frac{1}{2}$, $\theta(G_{19}) = 25$, $\theta(G_{26}) = 14\frac{1}{2}$ and $\theta(G_{27}) = 26\frac{1}{2}$;

(xiii) G_{13} is the improvement of G_{14}, G_{17}, G_{22}, G_{24}, G_{26}, G_{28} and G_{29} with $\theta(G_{14}) = 11\frac{1}{2}$, $\theta(G_{17}) = 8\frac{1}{4}$, $\theta(G_{22}) = 9$, $\theta(G_{24}) = 9\frac{1}{4}$, $\theta(G_{26}) = 14\frac{1}{2}$, $\theta(G_{28}) = 15$ and $\theta(G_{29}) = 15\frac{1}{4}$, ;

(xiv) G_{15} is the improvement of G_{16} and G_{30} with $\theta(G_{16}) = 6\frac{1}{4}$ and $\theta(G_{30}) = 8\frac{1}{8}$;

(xv) G_{16} is the improvement of G_{17}, G_{18}, G_{24}, G_{25}, G_{30}, G_{31} and G_{32} with $\theta(G_{17}) = 8\frac{1}{4}$, $\theta(G_{18}) = 12\frac{1}{4}$, $\theta(G_{24}) = 9\frac{1}{4}$, $\theta(G_{25}) = 13\frac{1}{4}$, $\theta(G_{30}) = 8\frac{1}{8}$, $\theta(G_{31}) = 10\frac{1}{8}$ and $\theta(G_{32}) = 14\frac{1}{8}$;

(xvi) G_{17} is the improvement of G_{18}, G_{24}, G_{29}, G_{31} and G_{33} with $\theta(G_{18}) = 12\frac{1}{4}$, $\theta(G_{24}) = 9\frac{1}{4}$, $\theta(G_{29}) = 15\frac{1}{4}$, $\theta(G_{31}) = 10\frac{1}{8}$ and $\theta(G_{33}) = 16\frac{1}{8}$;

(xvii) G_{22} is the improvement of G_{23}, G_{24}, G_{28}, G_{34} and G_{35} with $\theta(G_{23}) = 13$, $\theta(G_{24}) = 9\frac{1}{4}$, $\theta(G_{28}) = 15$, $\theta(G_{34}) = 12\frac{3}{4}$ and $\theta(G_{35}) = 16\frac{3}{4}$;

(xviii) G_{24} is the improvement of G_{25}, G_{29}, G_{31}, G_{34}, G_{35}, G_{36} and G_{37} with $\theta(G_{25}) = 13\frac{1}{4}$, $\theta(G_{29}) = 15\frac{1}{4}$, $\theta(G_{31}) = 10\frac{1}{8}$, $\theta(G_{34}) = 12\frac{3}{4}$, $\theta(G_{35}) = 16\frac{3}{4}$, $\theta(G_{36}) = 13\frac{1}{8}$ and $\theta(G_{37}) = 17\frac{1}{8}$;

(xix) G_{30} is the improvement of G_{31}, G_{32}, G_{38} and G_{39} with $\theta(G_{31}) = 10\frac{1}{8}$, $\theta(G_{32}) = 14\frac{1}{8}$, $\theta(G_{38}) = 12\frac{1}{16}$ and $\theta(G_{39}) = 16\frac{1}{16}$;

(xx) G_{31} is the improvement of G_{32}, G_{33}, G_{36}, G_{37}, G_{38}, G_{40} and G_{41} with $\theta(G_{32}) = 14\frac{1}{8}$, $\theta(G_{33}) = 16\frac{1}{8}$, $\theta(G_{36}) = 13\frac{1}{8}$, $\theta(G_{37}) = 17\frac{1}{8}$, $\theta(G_{38}) = 12\frac{1}{16}$, $\theta(G_{40}) = 14\frac{1}{16}$ and $\theta(G_{41}) = 18\frac{1}{16}$.

Hence, by Lemma 2.4 and the above arguments, we know (i) to (xiv) holds. Thus the proof is completed. □

Similarly to the proof of Theorem 3.1, we can obtain Theorems 3.2 and 3.3.

Theorem 3.2 *Let $G = K(n_1, n_2, n_3, n_4, n_5)$ be a complete 5-partite graph such that $n_1 + n_2 + n_3 + n_4 + n_5 = 5n + 2$ and $n_5 - n_1 \leqslant 4$. Define $\theta(G) = [\alpha(G, 6) - 3 \cdot 2^n - 2^{n-1} + 5]/2^{n-2}$. Then*

(i) $\theta(G) = 0$ if and only if $G = K(n, n, n, n+1, n+1)$;

(ii) $\theta(G) = 1$ if and only if $G = K(n-1, n, n+1, n+1, n+1)$;

(iii) $\theta(G) = 2$ if and only if $G = K(n, n, n, n, n+2)$;

(iv) $\theta(G) = 2\frac{1}{2}$ if and only if $G = K(n-2, n+1, n+1, n+1, n+1)$;

(v) $\theta(G) = 3$ if and only if $G = K(n-1, n, n, n+1, n+2)$;

(vi) $\theta(G) = 4$ if and only if $G = K(n-1, n-1, n+1, n+1, n+2)$;

(vii) $\theta(G) = 4\frac{1}{2}$ if and only if $G = K(n-2, n, n+1, n+1, n+2)$;

(viii) $\theta(G) = 6$ if and only if $G = K(n-1, n-1, n, n+2, n+2)$;

(ix) $\theta(G) = 6\frac{1}{2}$ if and only if $G = K(n-2, n, n, n+2, n+2)$;

(x) $\theta(G) = 7\frac{1}{2}$ if and only if $G = K(n-2, n-1, n+1, n+2, n+2)$;

(xi) $\theta(G) = 9$ if and only if $G = K(n-1, n, n, n, n+3)$;

(xii) $\theta(G) = 10$ if and only if $G = K(n-1, n-1, n, n+1, n+3)$;

(xiii) $\theta(G) = 11$ *if and only if* $G = K(n-2, n-2, n+2, n+2, n+2)$*;*

(xiv) $\theta(G) = 13$ *if and only if* $G = K(n-1, n-1, n-1, n+2, n+3)$*.*

Theorem 3.3 *Let* $G = K(n_1, n_2, n_3, n_4, n_5)$ *be a complete 5-partite graph such that* $n_1 + n_2 + n_3 + n_4 + n_5 = 5n + 3$ *and* $n_5 - n_1 \leqslant 4$*. Define* $\theta(G) = [\alpha(G, 6) - 2^{n+2} + 5]/2^{n-1}$*. Then*

(i) $\theta(G) = 0$ *if and only if* $G = K(n, n, n+1, n+1, n+1)$*;*

(ii) $\theta(G) = \frac{1}{2}$ *if and only if* $G = K(n-1, n+1, n+1, n+1, n+1)$*;*

(iii) $\theta(G) = 1$ *if and only if* $G = K(n, n, n, n+1, n+2)$*;*

(iv) $\theta(G) = 1\frac{1}{2}$ *if and only if* $G = K(n-1, n, n+1, n+1, n+2)$*;*

(v) $\theta(G) = 2\frac{1}{4}$ *if and only if* $G = K(n-2, n+1, n+1, n+1, n+2)$*;*

(vi) $\theta(G) = 2\frac{1}{2}$ *if and only if* $G = K(n-1, n, n, n+2, n+2)$*;*

(vii) $\theta(G) = 3$ *if and only if* $G = K(n-1, n-1, n+1, n+2, n+2)$*;*

(viii) $\theta(G) = 3\frac{1}{4}$ *if and only if* $G = K(n-2, n, n+1, n+2, n+2)$*;*

(ix) $\theta(G) = 4$ *if and only if* $G = K(n, n, n, n, n+3)$*;*

(x) $\theta(G) = 4\frac{1}{2}$ *if and only if* $G = K(n-1, n, n, n+1, n+3)$*;*

(xi) $\theta(G) = 4\frac{3}{4}$ *if and only if* $G = K(n-2, n-1, n+2, n+2, n+2)$*;*

(xii) $\theta(G) = 5$ *if and only if* $G = K(n-1, n-1, n+1, n+1, n+3)$*;*

(xiii) $\theta(G) = 6$ *if and only if* $G = K(n-1, n-1, n, n+2, n+3)$*;*

(xiv) $\theta(G) = 9\frac{1}{2}$ *if and only if* $G = K(n-1, n-1, n-1, n+3, n+3)$*.*

§4. **Chromatically Closed 5-Partite Graphs**

In this section, we obtained several χ-closed families of graphs from the graphs in Theorem 3.1 to 3.3.

Theorem 4.1 *The family of graphs* $\mathcal{K}^{-s}(n_1, n_2, n_3, n_4, n_5)$ *where* $n_1+n_2+n_3+n_4+n_5 = 5n+1$*,* $n_5 - n_1 \leqslant 4$ *and* $n_1 \geqslant s+5$ *is* χ*-closed.*

Proof By Theorem 3.1, there are 14 cases to consider. Denote each graph in Theorem 3.1 $(i), (ii), \cdots, (xiv)$ by $G_1, G_2, \cdots, G_{14}$, respectively. Suppose $H \sim G_i - S$. It suffices to show that $H \in \{G_i - S\}$. By Lemma 2.3, we know there exists a complete 5-partite graph $F = (p_1, p_2, p_3, p_4, p_5)$ such that $H = F - S'$ with $|S'| = s' = e(F) - e(G) + s \geqslant 0$.

Case 1. Let $G = G_1$ with $n \geqslant s+2$. In this case, $H \sim F - S \in \mathcal{K}^{-s}(n,n,n,n,n+1)$. By Lemma 2.5, we have

$$\alpha(G-S,6) = \alpha(G,6) + \alpha'(G-S) \text{ with } s \leqslant \alpha'(G-S) \leqslant 2^s - 1,$$
$$\alpha(F-S',6) = \alpha(F,6) + \alpha'(F-S') \text{ with } 0 \leqslant s' \leqslant \alpha'(F-S').$$

Hence,

$$\alpha(F-S',6) - \alpha(G-S,6) = \alpha(F,6) - \alpha(G,6) + \alpha'(F-S') - \alpha'(G-S).$$

By the definition, $\alpha(F,6) - \alpha(G,6) = 2^{n-2}(\theta(F) - \theta(G))$. By Theorem 3.1, $\theta(F) \geqslant 0$. Suppose $\theta(F) > 0$, then

$$\begin{aligned} \alpha(F-S',6) - \alpha(G-S,6) &\geqslant 2^{n-2} + \alpha'(F-S') - \alpha'(G-S) \\ &\geqslant 2^s + \alpha'(F-S') - 2^s + 1 \geqslant 1, \end{aligned}$$

contradicting $\alpha(F-S',6) = \alpha(G-S,6)$. Hence, $\theta(F) = 0$ and so $F = G$ and $s = s'$. Therefore, $H \in \mathcal{K}^{-s}(n,n,n,n,n+1)$.

Case 2. Let $G = G_2$ with $n \geqslant s+3$. In this case, $H \sim F - S \in \mathcal{K}^{-s}(n-1,n,n,n+1,n+1)$. By Lemma 2.5, we have

$$\alpha(G-S,6) = \alpha(G,6) + \alpha'(G-S) \text{ with } s \leqslant \alpha'(G-S) \leqslant 2^s - 1,$$
$$\alpha(F-S',6) = \alpha(F,6) + \alpha'(F-S') \text{ with } 0 \leqslant s' \leqslant \alpha'(F-S').$$

Hence,

$$\alpha(F-S',6) - \alpha(G-S,6) = \alpha(F,6) - \alpha(G,6) + \alpha'(F-S') - \alpha'(G-S).$$

By the definition, $\alpha(F,6) - \alpha(G,6) = 2^{n-2}(\theta(F) - \theta(G))$. Suppose $\theta(F) \neq \theta(G)$. Then, we consider two subcases.

Subcase 2.1 $\theta(F) < \theta(G)$. By Theorem 3.1, $F = G_1$ and $H = G_1 - S' \in \{G_1 - S'\}$. However, $G - S \notin \{G_1 - S'\}$ since by Case (i) above, $\{G_1 - S'\}$ is χ-closed, a contradiction.

Subcase 2.2 $\theta(F) > \theta(G)$. By Theorem 3.1, $\alpha(F,6) - \alpha(G,6) \geqslant 2^{n-2}$. So,

$$\begin{aligned} \alpha(F-S',6) - \alpha(G-S,6) &\geqslant 2^{n-2} + \alpha'(F-S') - \alpha'(G-S) \\ &\geqslant 2^s + \alpha'(F-S') - 2^s + 1 \geqslant 1, \end{aligned}$$

contradicting $\alpha(F-S',6) = \alpha(G-S,6)$. Hence, $\theta(F) - \theta(G) = 0$ and so $F = G$ and $s = s'$. Therefore, $H \in \mathcal{K}^{-s}(n-1,n,n,n+1,n+1)$.

Using Table 1, we can prove (iii) to (xiv) in a similar way. This completes the proof. □

Similarly, we can prove Theorems 4.2 and 4.3.

Theorem 4.2 *The family of graphs* $\mathcal{K}^{-s}(n_1,n_2,n_3,n_4,n_5)$ *where* $n_1+n_2+n_3+n_4+n_5 = 5n+2$, $n_5 - n_1 \leqslant 4$ *and* $n_1 \geqslant s+6$ *is* χ*-closed.*

Theorem 4.3 *The family of graphs $\mathcal{K}^{-s}(n_1,n_2,n_3,n_4,n_5)$ where $n_1+n_2+n_3+n_4+n_5=5n+3$, $n_5-n_1\leqslant 4$ and $n_1\geqslant s+6$ is χ-closed.*

§5. Chromatically Unique 5-Partite Graphs

The following results give several families of chromatically unique complete 5-partite graphs having $5n+1$ vertices with a set S of s edges deleted where the deleted edges induce a star $K_{1,s}$ and a matching sK_2, respectively.

Theorem 5.1 *The graphs $K_{i,j}^{-K_{1,s}}(n_1,n_2,n_3,n_4,n_5)$ where $n_1+n_2+n_3+n_4+n_5=5n+1$, $n_5-n_1\leqslant 4$ and $n_1\geqslant s+5$ are χ-unique for $1\leqslant i\neq j\leqslant 5$.*

Proof By Theorem 3.1, there are 14 cases to consider. Denote each graph in Theorem 3.1 $(i),(ii),\cdots,(xiv)$ by $G_1,G_2,\cdots,G_{14}$, respectively. The proof for each graph obtained from G_i $(i=1,2,\cdots,14)$ is similar, so we only give the detail proof for the graphs obtained from G_2 below.

By Lemma 2.5 and Case 2 of Theorem 4.1, we know that $K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1)$ $=\{K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1)|(i,j)\in\{(1,2),(2,1),(1,4),(4,1),(2,3),(2,4),(4,2),(4,5)\}$ is χ-closed for $n\geqslant s+3$. Note that

$$t(K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1))=t(G_2)-s(3n+2)\text{ for }(i,j)\in\{(1,2),(2,1)\},$$
$$t(K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1))=t(G_2)-s(3n+1)\text{ for }(i,j)\in\{(1,4),(4,1),(2,3)\},$$
$$t(K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1))=t(G_2)-3sn\text{ for }(i,j)\in\{(2,4),(4,2)\},$$
$$t(K_{4,5}^{-K_{1,s}}(n-1,n,n,n+1,n+1))=t(G_2)-s(3n-1).$$

By Lemmas 2.2 and 2.6, we conclude that $\sigma(K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1))\neq\sigma(K_{j,i}^{-K_{1,s}}(n-1,n,n,n+1,n+1))$ for each $(i,j)\in\{(1,2),(1,4),(2,4)\}$. We now show that $K_{2,3}^{-K_{1,s}}(n-1,n,n,n+1,n+1)$ and $K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1)$ for $(i,j)\in\{(1,4),(4,1)\}$ are not χ-equivalent. We have

$$Q(K_{2,3}^{-K_{1,s}}(n-1,n,n,n+1,n+1))=Q(G_2)-s(n-1)^2+\binom{s}{2}+s\left[\binom{n-1}{2}+2\binom{n+1}{2}\right],$$

$$Q(K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1))=Q(G_2)-sn(n-2)+\binom{s}{2}+s\left[2\binom{n}{2}+\binom{n+1}{2}\right]$$

for $(i,j)\in\{(1,4),(4,1)\}$ with

$$Q\Big(K_{2,3}^{-K_{1,s}}(n-1,n,n,n+1,n+1)\Big)-Q\Big(K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1)\Big)=0$$

since $s_{ij}=0$ if $(i,j)\neq\{(1,4),(4,1),(2,3)\}$. We also obtain

$$\begin{aligned}K(K_{2,3}^{-K_{1,s}}(n-1,n,n,n+1,n+1)) &= K(G_2)-s(3n^2+2n-1);\\ K(K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1)) &= K(G_2)-s(3n^2+2n)\end{aligned}$$

for $(i,j) \in \{(1,4),(4,1)\}$ with

$$K\Big(K_{2,3}^{-K_{1,s}}(n-1,n,n,n+1,n+1)\Big) - K\Big(K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1)\Big) = s$$

since $s_{ij} = 0$ if $(i,j) \neq \{(1,4),(4,1),(2,3)\}$. This means that $2K(K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1)) - Q(K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1)) \neq 2K(K_{2,3}^{-K_{1,s}}(n-1,n,n,n+1,n+1)) - Q(K_{2,3}^{-K_{1,s}}(n-1,n,n,n+1,n+1))$ for $(i,j) \in \{(1,4),(4,1)\}$, contradicting Lemma 2.1. Hence, $K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1)$ is χ-unique where $n \geqslant s+3$ for $1 \leqslant i \neq j \leqslant 5$. The proof is thus complete. □

Theorem 5.2 *The graphs $K_{1,2}^{-sK_2}(n_1,n_2,n_3,n_4,n_5)$ where $n_1+n_2+n_3+n_4+n_5 = 5n+1$, $n_5 - n_1 \leqslant 4$ and $n_1 \geqslant s+5$ are χ-unique.*

Proof By Theorem 3.1, there are 14 cases to consider. Denote each graph in Theorem 3.1 $(i),(ii),\cdots,(xiv)$ by $G_1,G_2,\cdots,G_{14}$, respectively. For a graph $K(p_1,p_2,p_3,p_4,p_5)$, let $S = \{e_1,e_2,\cdots,e_s\}$ be the set of s edges in $E(K(p_1,p_2,p_3,p_4,p_5))$ and let $t(e_i)$ denote the number of triangles containing e_i in $K(p_1,p_2,p_3,p_4,p_5)$. The proofs for each graph obtained from G_i $(i=1,2,\cdots,14)$ are similar, so we only give the proof of the graph obtained from G_1 and G_2 as follows.

Suppose $H \sim G = K_{1,2}^{-sK_2}(n,n,n,n,n+1)$ for $n \geqslant s+2$. By Theorem 4.1 and Lemma 2.1, $H \in \mathcal{K}^{-s}(n,n,n,n,n+1)$ and $\alpha'(H) = \alpha'(G) = s$. Let $H = F-S$ where $F = K(n,n,n,n,n+1)$. Clearly, $t(e_i) \leqslant 3n+1$ for each $e_i \in S$. So,

$$t(H) \geqslant t(F) - s(3n+1),$$

with equality holds only if $t(e_i) = 3n+1$ for all $e_i \in S$. Since $t(H) = t(G) = t(F) - s(3n+1)$, the equality above holds with $t(e_i) = 3n+1$ for all $e_i \in S$. Therefore each edge in S has an end-vertex in V_i and another end-vertex in V_j $(1 \leqslant i < j \leqslant 4)$. Moreover, S must induce a matching in F. Otherwise, equality does not hold or $\alpha'(H) > s$. By Lemma 2.8, we obtain

$$Q(G) = Q(F) - s(n-1)^2 + \binom{s}{2} + s\left[2\binom{n}{2} + \binom{n+1}{2}\right]$$

whereas

$$\begin{aligned} Q(H) &= Q(F) - s(n-1)^2 + \binom{s}{2} - s_{12}(s_{13}+s_{14}+s_{23}+s_{24}+s_{34}) \\ &\quad -s_{13}(s_{14}+s_{23}+s_{24}+s_{34}) - s_{14}(s_{23}+s_{24}+s_{34}) - s_{23}(s_{24}+s_{34}) - s_{24}s_{34} \\ &\quad +s\left[2\binom{n}{2} + \binom{n+1}{2}\right] + s_{12}s_{34} + s_{13}s_{24} + s_{14}s_{23} \\ &= Q(G) - s_{12}(s_{13}+s_{14}+s_{23}+s_{24}) - s_{13}(s_{14}+s_{23}+s_{34}) - s_{14}(s_{24}+s_{34}) \\ &\quad -s_{23}(s_{24}+s_{34}) - s_{24}s_{34}. \end{aligned}$$

Moreover, $K(G) = K(F) - s(3n^2+2n)$ whereas

$$\begin{aligned} K(H) &= K(F) - s(3n^2+2n) + s_{12}s_{34} + s_{13}s_{24} + s_{14}s_{23} \\ &= K(G) + s_{12}s_{34} + s_{13}s_{24} + s_{14}s_{23}. \end{aligned}$$

Hence,

$$\begin{aligned} 2K(H)-Q(H) &= 2K(G)-Q(G)+2(s_{12}s_{34}+s_{13}s_{24}+s_{14}s_{23})+\\ &\quad s_{12}(s_{13}+s_{14}+s_{23}+s_{24})+s_{13}(s_{14}+s_{23}+s_{34})+s_{14}(s_{24}+s_{34})+\\ &\quad s_{23}(s_{24}+s_{34})+s_{24}s_{34}, \end{aligned}$$

and that $2K(H)-Q(H)=2K(G)-Q(G)$ if and only if $s=s_{ij}$ for $1\leqslant i<j\leqslant 4$. Therefore, we have $\langle S\rangle\cong sK_2$ with $H\cong G$.

Suppose $H\sim G=K_{1,2}^{-sK_2}(n-1,n,n,n+1,n+1)$ for $n\geqslant s+3$. By Theorem 4.1 and Lemma 2.1, $H\in\mathcal{K}^{-s}(n-1,n,n,n+1,n+1)$ and $\alpha'(H)=\alpha'(G)=s$. Let $H=F-S$ where $F=K(n-1,n,n,n+1,n+1)$. Clearly, $t(e_i)\leqslant 3n+2$ for each $e_i\in S$. So,

$$t(H)\geqslant t(F)-s(3n+2),$$

with equality holds only if $t(e_i)=3n+2$ for all $e_i\in S$. Since $t(H)=t(G)=t(F)-s(3n+2)$, the equality above holds with $t(e_i)=3n+2$ for all $e_i\in S$. Therefore each edge in S has an end-vertex in V_1 and another end-vertex in V_j ($2\leqslant j\leqslant 3$). Moreover, S must induce a matching in F. Otherwise, equality does not hold or $\alpha'(H)>s$. By Lemma 2.8, we obtain

$$Q(G)=Q(F)-s(n-1)(n-2)+\binom{s}{2}+s\left[\binom{n}{2}+2\binom{n+1}{2}\right]$$

whereas

$$\begin{aligned} Q(H) &= Q(F)-s(n-1)(n-2)+\binom{s}{2}-s_{12}s_{13}+s\left[\binom{n}{2}+2\binom{n+1}{2}\right]\\ &\leqslant Q(G), \end{aligned}$$

and the equality holds if and only if $s=s_{1j}$ ($2\leqslant j\leqslant 3$). Moreover, $K(G)=K(H)=K(F)-s(3n^2+4n+1)$. Hence, $2K(G)-Q(G)\neq 2K(H)-Q(H)$ and the equality holds if and only if $\langle S\rangle\cong sK_2$ with $H\cong G$. Thus the proof is complete. □

Similarly to the proofs of Theorems 5.1 and 5.2, we can prove Theorems 5.3 to 5.6 following.

Theorem 5.3 *The graphs* $K_{i,j}^{-K_{1,s}}(n_1,n_2,n_3,n_4,n_5)$ *where* $n_1+n_2+n_3+n_4+n_5=5n+2$, $n_5-n_1\leqslant 4$ *and* $n_1\geqslant s+6$ *are* χ*-unique for* $1\leqslant i\neq j\leqslant 5$.

Theorem 5.4 *The graphs* $K_{i,j}^{-K_{1,s}}(n_1,n_2,n_3,n_4,n_5)$ *where* $n_1+n_2+n_3+n_4+n_5=5n+3$, $n_5-n_1\leqslant 4$ *and* $n_1\geqslant s+6$ *are* χ*-unique for* $1\leqslant i\neq j\leqslant 5$.

Theorem 5.5 *The graphs* $K_{1,2}^{-sK_2}(n_1,n_2,n_3,n_4,n_5)$ *where* $n_1+n_2+n_3+n_4+n_5=5n+2$, $n_5-n_1\leqslant 4$ *and* $n_1\geqslant s+6$ *are* χ*-unique.*

Theorem 5.6 *The graphs* $K_{1,2}^{-sK_2}(n_1,n_2,n_3,n_4,n_5)$ *where* $n_1+n_2+n_3+n_4+n_5=5n+3$, $n_5-n_1\leqslant 4$ *and* $n_1\geqslant s+6$ *are* χ*-unique.*

Remark 5.7 This paper generalized the results and solved the open problems in [9,10,11].

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The Order of the Sandpile Group of Infinite Complete Expansion Regular Graphs

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Abstract: The sandpile group or critical group of a graph is an Abelian group whose order is the number of spanning trees of the graph. In the paper, the order of the sandpile group of infinite complete expansion regular graphs is obtained.

Key Words: Sandpile group, expansion graph, infinite complete expansion graph.

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§1. Introduction

The sandpile group or critical group $K(G)$ of a graph G is an isomorphism invariant that comes in the form of a finite Abelian group; its order the *complexity* $\kappa(G)$, that is, the number of spanning trees in G. The interested reader can find standard results on the subject in [2,Chapter 13] and in [3,4].

We explore here the order of sandpile group of infinite expansion transformation graph. The concept of expansion transformation graph of a graph was given by [5](Fig.1 is a simple example), and complete expansion graph of a graph G is the special expansion graphs of G(see Fig.1), that is the line of subdivision of G. *Subdivision graph* sd$G$, obtained from placing a new vertex in the center of every edge of G, and the line of sd$G$ is complete expansion graph, we denote it as $EXP(G)$.

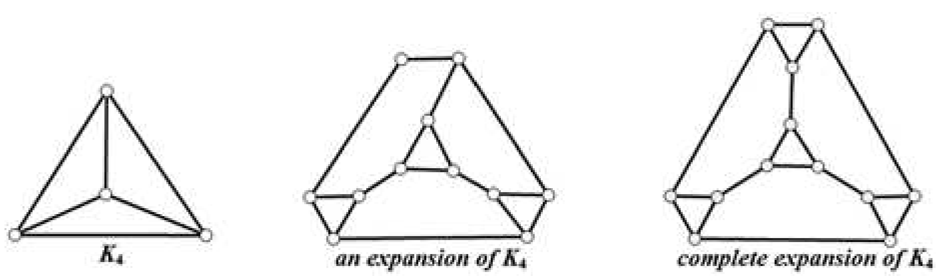

Fig.1 K_4**, an expansion and complete expansion graph of** K_4

Hence graph G and $EXP(G)$ are both special graphs of the expansions graph of G. We said G be an ordinary expansion of G. The complete expansion of $EXP(G)$, we denote as $EXP^2(G)$,

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that is, $EXP^2(G) = EXP(EXP(G))$, similarly, $m \in \mathbb{N}, EXP^m(G) = EXP(EXP^{m-1}(G))$. We call $\{EXP^m(G)\},\ m = 1, 2, \cdots$, be an infinite complete expansion of graph G, simply denoted by ICEG.

Let $\beta(G)$ denote the number of linearly independent elements in the cycle space of G. The present paper refer to following results [6,7].

lemma 1.1(Sachs, Cvetković) *Let G be a connected with lineG regular.*

If G is d-regular, then

$$\kappa(lineG) = d^{\beta(G)-2}2^{\beta(G)}. \tag{1.1}$$

If G is bipartite and (d_1, d_2)-semiregular with bipartition V_1, V_2, then

$$\kappa(lineG) = \frac{(d_1 + d_2)^{\beta(G)}}{d_1 d_2}\frac{(}{d_1}d_2)^{|V_1|-|V_2|}\kappa(G). \tag{1.2}$$

Lemma 1.2 *Let G be a connected graph. Then*

$$\kappa(sdG) = 2^{\beta(G)}\kappa(G). \tag{1.3}$$

These results suggests some close relationship between the sandpile group $K(G)$ and K(lineG) in either of these situations.

§2. Main Results and Proofs

Theorem 2.1 *Let G be a k- regular graph with n vertices, ε edges, then $\forall m \in \mathbb{N}, m \geq 1$,*

(1) *$EXP^m(G)$ have $2k^{m-1}\varepsilon$ vertices and εk^m edges;*

(2) *$SdEXP^m(G)$ have $2k^{m-1}\varepsilon + k^m\varepsilon$ vertices and $2k^m\varepsilon$ edges.*

Proof We use n, ε to denote $n(G), \varepsilon(G)$ in the following, and show that the results by induction for m. Since subdivision graph sdG obtained from G placing a new vertex in the center of every edge of G, hence $n(sdG) = \varepsilon + n, \varepsilon(sdG) = nk = 2\varepsilon$, and so the $linesdG$ have 2ε vertices and $k\varepsilon$ edges, that is, $n(EXP(G)) = 2\varepsilon, \varepsilon(EXP(G)) = \varepsilon k$, the result is true for $m = 1$.

Assume that result is true for $m-1$, that is, $n(EXP^{m-1}(G)) = 2k^{m-2}\varepsilon, \varepsilon(EXP^{m-1}(G)) = \varepsilon k^{m-1}$. Then subdivision graph $sdEXP^{m-1}(G)$ obtained from $EXP^{m-1}(G)$ placing a new vertex in the center of every edge of $EXP^{m-1}(G)$, hence $n(sdEXP^{m-1}(G)) = \varepsilon k^{m-1} + 2k^{m-2}\varepsilon, \varepsilon(sdG) = \varepsilon k^{m-1}$. The details of the direct proof refer to the table below.

ICEG of k regular graph G_0				
	graphs	*vertex-number*	*edge-number*	*Remarks*
0	G_0	n	ε	
0	sdG_0	$n+\varepsilon$	2ε	subd G_0
1	$G_1 = EXP(G_0)$	2ε	εk	linesubd G_0
1	sd$G_1 = sdEXP(G_0)$	$2\varepsilon + \varepsilon k$	$2\varepsilon k$	subd G_1
2	$G_2 = EXP^2(G_0)$	$2\varepsilon k$	εk^2	linesubd G_1
2	sd$G_2 = sdEXP^2(G_0)$	$2\varepsilon k + \varepsilon k^2$	$2\varepsilon k^2$	subdG_2
3	$G_3 = EXP^3(G_0)$	$2\varepsilon k^2$	εk^3	linesubd G_2
3	sd$G_3 = sdEXP^3(G_0)$	$2\varepsilon k^2 + \varepsilon k^3$	$2\varepsilon k^3$	subd G_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m-1$	sd$G_{m-1} = sdEXP^{m-1}(G_0)$	$2\varepsilon k^{m-2} + \varepsilon k^{m-1}$	$2\varepsilon k^{m-1}$	subd G_{m-1}
m	$G_m = EXP^m(G_0)$	$2\varepsilon k^{m-1}$	εk^m	linesubd G_{m-1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Remark *subdG* denotes subdivision of graph G; *linesubdG* denotes the line graph of the subdivision of the graph G. □

Theorem 2.2 *Let G be a k- regular graph with n vertices, ε edges, then $\forall m \in \mathbb{N}, m \geq 1$, the order of sandpile group of $EXP^m(G)$ equals to*

$$2^{m(\omega-1)} \cdot (2+k)^{\frac{k^{m-1}(k-2)+1}{k-1}\varepsilon - n + m\omega} \cdot k^{\frac{k^{m-1}(k-2)+1}{k-1}\varepsilon - n - m}\kappa(G). \tag{2.1}$$

Proof The proof is by mathematical induction on m. If $m = 1$, by (1.3) $\kappa(sd(G)) = 2^{\varepsilon-n+\omega} \cdot \kappa(G)$, and by (1.2)

$$\kappa(linesd(G)) = \frac{(2+k)^{\beta(sdG)}}{2k} \cdot (\frac{k}{2})^{\varepsilon-n} \cdot \kappa(sd(G)) \tag{2.2}$$

Put $\kappa(linesd(G)) = \kappa(EXP(G))$ and $\beta(sdG) = 2\varepsilon - (n+\varepsilon) + \omega = \varepsilon - n + \omega$ and $\kappa(sd(G)) = 2^{\varepsilon-n+\omega}$ into the (2.2),

$$\kappa(EXP(G)) = \frac{(2+k)^{\varepsilon-n+\omega}}{2k} \cdot (\frac{k}{2})^{\varepsilon-n} \cdot 2^{\varepsilon-n+\omega}\kappa(G),$$

that is

$$\kappa(EXP(G)) = 2^{\omega-1}(2+k)^{\varepsilon-n+\omega} \cdot k^{\varepsilon-n-1}\kappa(G), \tag{2.3}$$

hence (2.1) is true for $m = 1$.

Now assume (2.1) be true for $m-1$. Since $\kappa(EXP^m(G)) = \kappa(linesdG_{m-1})$, and

$$\kappa(linesdG_{m-1}) = \frac{(2+k)^{\beta(sdG_{m-1})}}{2k} \cdot (\frac{k}{2})^{\varepsilon(G_{m-1})-n(G_{m-1})} \cdot \kappa(sdG_{m-1}), \tag{2.4}$$

we have

$$\beta(sdG_{m-1}) = \varepsilon(G_{m-1}) - n(G_{m-1}) + \omega,$$
$$\varepsilon(G_{m-1}) = 2\varepsilon k^{m-1}, n(G_{m-1}) = 2\varepsilon k^{m-2} + \varepsilon k^{m-1},$$

and by inductive hypothesis $\kappa(sdG_{m-1}) = 2^{\beta(G_{m-1})} \cdot \kappa(G_{m-1})$,

$$\kappa(G_{m-1}) = 2^{(m-1)(\omega-1)} \cdot (2+k)^{\frac{k^{m-2}(k-2)+1}{k-1}\varepsilon - n + (m-1)\omega} \cdot k^{\frac{k^{m-2}(k-2)+1}{k-1}\varepsilon - n - (m-1)}\kappa(G).$$

Substitute all of above into the (2.4), we get that

$$\kappa(EXP^m(G)) = \frac{(2+k)^{\varepsilon k^{m-1} - 2\varepsilon k^{m-2} + \omega}}{2k} \cdot (\frac{k}{2})^{\varepsilon k^{m-1} - 2\varepsilon k^{m-2}} \cdot 2^{\beta(G_{m-1})} \cdot \kappa(G_{m-1}),$$

that is,

$$\kappa(EXP^m(G)) = 2^{m(\omega-1)} \cdot (2+k)^{\frac{k^{m-1}(k-2)+1}{k-1}\varepsilon - n + m\omega} \cdot k^{\frac{k^{m-1}(k-2)+1}{k-1}\varepsilon - n - m}\kappa(G). \qquad \square$$

Corollary 2.1 *Let G be a k- regular graph with n vertices, ε edges, if G is connected graph, then $\forall m \in \mathbb{N}, m \geq 1$*

$$\kappa(EXP^m(G)) = (2+k)^{\frac{k^{m-1}(k-2)+1}{k-1}\varepsilon - n + m} \cdot k^{\frac{k^{m-1}(k-2)+1}{k-1}\varepsilon - n - m}\kappa(G). \tag{2.5}$$

Specially, if $k = 2$, then

$$\kappa(EXP^m(G)) = 2^m \cdot \kappa(G) \tag{2.6}$$

and if $k = 3$, then

$$\kappa(EXP^m(G)) = 5^{\frac{3^{m-1}+1}{2} \cdot \varepsilon - n + m} \cdot 3^{\frac{3^{m-1}+1}{2} \cdot \varepsilon - n - m}\kappa(G), \tag{2.7}$$

if $m = 1$, then

$$\kappa(EXP(G)) = (2+k)^{\varepsilon - n + 1} \cdot k^{\varepsilon - n - 1} \cdot \kappa(G), \tag{2.8}$$

if $m = 2$, then

$$\kappa(EXP^2(G)) = (2+k)^{(k-1)\varepsilon - n + 2} \cdot k^{(k-1)\varepsilon - n - 2} \cdot \kappa(G). \tag{2.9}$$

Proof Let $\omega = 1$ in (2.1), we have (2.5) at once; and $k = 2, 3$ in (2.5) obtained (2.6) and (2.7); $m = 1, 2$ in (2.5) obtained (2.8) and (2.9). $\square$

§3. Examples

Example 3.1 Let G be a loop, then $\kappa(G) = 1$, and $EXP(G) = C_2$.[C_t denotes t-cycle] By (2.6), the order of sandpile group of $EXP(G)$, that is,

$$\kappa(C_2) = \kappa(EXP(G)) = 2.$$

Similarly

$$EXP^2(G) = C_{2^2}, \kappa(C_4) = \kappa(EXP^2(G)) = 2^2;$$

$$\cdots\cdots;$$

$$EXP^m(G) = C_{2^m}, \kappa(C_{2^m}) = \kappa(EXP^m(G)) = 2^m.$$

Example 3.2 Let θ be a θ-graph, then $\kappa(\theta) = 3$, and $EXP(\theta)$ is a Prism(see Fig.2). Then by (2.7), we have

$$\kappa(EXP(\theta)) = 5^{3-2+1} \cdot 3^{3-2-1} \cdot 3 = 75; \tag{3.1}$$

and

$$\kappa(EXP^2(\theta)) = 5^{\frac{3^{2-1}+1}{2}\cdot 3-2+2} \cdot 3^{\frac{3^{2-1}+1}{2}\cdot 3-2-2} \cdot 3 = 421875, \tag{3.2}$$

or by (2.8) , (2.1) and graph $EXP(\theta)$, we also have

$$\begin{aligned}
&\kappa(EXP^2(\theta)) \\
&= (2+k)^{\varepsilon(EXP(\theta))-n(EXP(\theta))+1} \cdot k^{\varepsilon(EXP(\theta))-n(EXP(\theta))-1} \cdot \kappa(EXP(\theta)) \\
&= 5^{9-6+1} \cdot 3^{9-6-1} \cdot 75 = 421875.
\end{aligned}$$

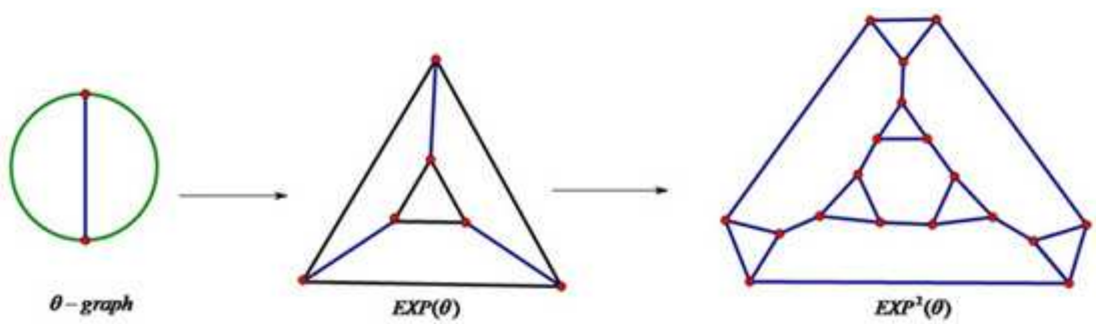

Fig.2 **θ-graph , 1th and 2th complete expansion graphs of the θ-graph**

Generally, if G is a multiplicity k's edges graph, that is, G have 2 vertices and k edges no loop connected graph, then by (2.5),

$$\kappa(EXP^m(G)) = (2+k)^{\frac{k^{m-1}(k-2)+1}{k-1}\cdot k-2+m} \cdot k^{\frac{k^{m+1}-2k^m+1}{k-1}-m}. \tag{3.3}$$

By (2.1),

$$m = 1 \Longrightarrow \kappa(EXP(G)) = (2+k)^{k-2} \cdot k^{k-1}; \tag{3.4}$$

and

$$m = 2 \Longrightarrow \kappa(EXP^2(G)) = (2+k)^{(k-1)k} \cdot k^{(k-1)k-3}. \tag{3.5}$$

Example 3.3 Let G be K_4, then $\kappa(K_4) = 4^2$, and $EXP(K_4)$ as Figure 3. By (2.5), we have

$$\kappa(EXP(K_4)) = 5^{6-4+1} \cdot 3^{6-4-1}\kappa(K_4) = 5^3 \cdot 3 \cdot 4^2 = 6000 \tag{3.6}$$

and by (2.9)

$$\kappa(EXP^2(K_4)) = (2+3)^{6(3-1)-4+2} \cdot 3^{6(3-1)-4-2} \cdot 4^2 = 11390625 \times 10^4,$$

or by (2.8) and (3.6) we have

$$\begin{aligned}&\kappa(EXP^2(K_4))\\&=(2+k)^{\varepsilon(EXP(K_4))-n(EXP(K_4))+1}\cdot k^{\varepsilon(EXP(K_4))-n(EXP(K_4))-1}\cdot\kappa(EXP(K_4))\\&=5^{6+1}\cdot 3^{6-1}\cdot 6000=11390625\times 10^4.\end{aligned}$$

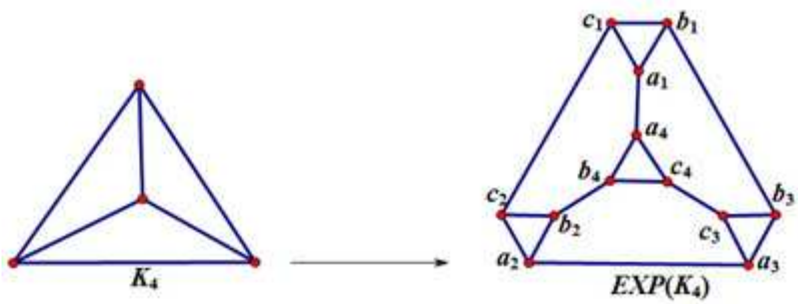

Fig.3 K_4 **and its expansion**

By Example 3.3 we have the following conclusion.

Proposition 3.1 *The order of sandpile group of Cayley graph* $Cay(A_4, \{(12),(123),(132)\})$ *is* 6000.

Proof Since $EXP(K_4) = Cay(A_4, \{(12),(123),(132)\})$, by[6], the proof is finished. □

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Solution of a Conjecture on Skolem Mean Graph of Stars $K_{1,l}\bigcup K_{1,m}\bigcup K_{1,n}$

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Abstract: In this paper, we prove a conjecture that the three stars $K_{1,l}\bigcup K_{1,m}\bigcup K_{1,n}$ is a skolem mean graph if $|m-n|<4+l$ for integers $l,m\geq 1$ and $l\leq m<n$.

Key Words: Smarandachely edge m-labeling f_S^*, Smarandachely super m-mean graph, skolem mean labeling, Skolem mean graph, star.

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§1. Introduction

All graphs in this paper are finite, simple and undirected. Terms not defined here are used in the sense of Harary [4]. A vertex labeling of G is an assignment $f:V(G)\to\{1,2,3,\ldots,p+q\}$ be an injection. For a vertex labeling f, the induced Smarandachely edge m-labeling f_S^* for an edge $e=uv$, an integer $m\geq 2$ is defined by $f_S^*(e)=\left\lceil\dfrac{f(u)+f(v)}{m}\right\rceil$. Then f is called a Smarandachely super m-mean labeling if $f(V(G))\cup\{f^*(e):e\in E(G)\}=\{1,2,3,\ldots,p+q\}$. Particularly, in the case of $m=2$, we know that

$$f^*(e)=\begin{cases}\dfrac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even;}\\ \dfrac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd.}\end{cases}$$

Such a labeling is usually called a mean labeling. A graph that admits a Smarandachely super mean m-labeling is called a Smarandachely super m-mean graph, particularly, a skolem mean graph if $m=2$ in [1]. It was proved that any path is a skolem mean graph, $K_{1,m}$ is not a skolem mean graphif $m\geq 4$, and the two stars $K_{1,m}\bigcup K_{1,n}$ is a skolem mean graph if and only if $|m-n|\leq 4$. In [2], it was proved that the three star $K_{1,l}\bigcup K_{1,m}\bigcup K_{1,n}$ is a skolem mean graph if $|m-n|=4+l$ for $l=1,2,3,\cdots$, $m=1,2,3,\cdots$ and $\leq m<n$. It is also shown in [2] that the three star $K_{1,l}\bigcup K_{1,m}\bigcup K_{1,n}$ is not a skolem mean graph if $|m-n|>4+l$ for $l=1,2,3,\cdots$, $m=1,2,3,\cdots$, $n\geq l+m+5$ and $l\leq m<n$, the four star $K_{1,l}\bigcup K_{1,l}\bigcup K_{1,m}\bigcup K_{1,n}$ is a skolem mean graph if $|m-n|=4+2l$ for $l=2,3,4,\cdots$, $m=2,3,4,\cdots$, $n=2l+m+4$ and $l\leq m<n$; the four star $K_{1,l}\bigcup K_{1,l}\bigcup K_{1,m}\bigcup K_{1,n}$ is not a skolem mean graph if $|m-n|>4+2l$ for $l=2,3,4,\cdots$, $m=2,3,4,\cdots$, $n\geq 2l+m+5$ and $l\leq m<n$; the four star $K_{1,1}\bigcup K_{1,1}\bigcup K_{1,m}\bigcup K_{1,n}$ is a skolem mean graph if $|m-n|=7$ for

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$m = 1, 2, 3, \cdots$, $n = m + 7$, $1 \leq m < n$, and the four star $K_{1,1} \bigcup K_{1,1} \bigcup K_{1,m} \bigcup K_{1,n}$ is not a skolem mean graph if $|m - n| > 7$ for $m = 1, 2, 3, \cdots$, $n \geq m + 8$ and $1 \leq m < n$. In [3], the condition for a graph to be skolem mean is that $p \geq q + 1$.

§2. Main Theorem

Definition 2.1 *The three star is the disjoint union of $K_{1,l}$, $K_{1,m}$ and $K_{1,n}$ for integers $l, m, n \geq 1$. Such a graph is denoted by $K_{1,l} \bigcup K_{1,m} \bigcup K_{1,n}$.*

Theorem 2.2 *If $l \leq m < n$, the three star $K_{1,l} \bigcup K_{1,m} \bigcup K_{1,n}$ is a skolem mean graph if $|m - n| < 4 + l$ for integers $l, m \geq 1$.*

Proof Consider the graph $G = K_{1,l} \bigcup K_{1,m} \bigcup K_{1,n}$. Let $\{u\} \bigcup \{u_i : 1 \leq i \leq l\}$, $\{v\} \bigcup \{v_j : 1 \leq j \leq m\}$ and $\{w\} \bigcup \{w_k : 1 \leq k \leq n\}$ be the vertices of G. Then G has $l + m + n + 3$ vertices and $l + m + n$ edges. We have $V(G) = \{u, v, w\} \bigcup \{u_i : 1 \leq i \leq l\} \bigcup \{v_j : 1 \leq j \leq m\} \bigcup \{w_k : 1 \leq k \leq n\}$. The proof id divided into four cases following.

Case 1 Let $l \leq m < n$ where $n = l + m + 3$ for integers $l, m \geq 1$. We prove such graph G is a skolem mean graph. The required vertex labeling $f : V(G) \rightarrow \{1, 2, 3, \cdots, l + m + n + 3\}$ is defined as follows:

$$\begin{aligned}
&f(u) = 1, \quad f(v) = 3;\\
&f(w) = l + m + n + 3;\\
&f(u_i) = 2i + 3 \text{ for } 1 \leq i \leq l;\\
&f(v_j) = 2l + 2j + 3 \text{ for } 1 \leq j \leq m;\\
&f(w_k) = 2k \text{ for } 1 \leq k \leq n - 1 \text{ and}\\
&f(w_n) = l + m + n + 2.
\end{aligned}$$

The corresponding edge labels are as follows:

The edge labels of uu_i is $i + 2$ for $1 \leq i \leq l$, vv_j is $l + j + 3$ for $1 \leq j \leq m$ and ww_k is $\dfrac{2k + l + m + n + 3}{2}$ for $1 \leq k \leq n - 1$. Also, the edge label of ww_n is $l + m + n + 3$. Therefore, the induced edge labels of G are distinct. Hence G is a skolem mean graph.

Case 2 Let $l \leq m < n$ where $n = l + m + 2$ for integers $l, m \geq 1$. We prove that G is a skolem mean graph. The required vertex labeling $f : V(G) \rightarrow \{1, 2, 3, \cdots, l + m + n + 3\}$ is defined as follows:

$$\begin{aligned}
&f(u) = 1; \; f(v) = 2; \; f(w) = l + m + n + 3;\\
&f(u_i) = 2i + 2 \text{ for } 1 \leq i \leq l;\\
&f(v_j) = 2l + 2j + 2 \text{ for } 1 \leq j \leq m;\\
&f(w_k) = 2k + 1 \text{ for } 1 \leq k \leq n - 1 \text{ and}\\
&f(w_n) = l + m + n + 2.
\end{aligned}$$

The corresponding edge labels are as follows:

The edge labels of uu_i is $i+2$ for $1 \le i \le l$; vv_j is $l+j+2$ for $1 \le j \le m$ and ww_k is $\dfrac{2k+l+m+n+4}{2}$ for $1 \le k \le n-1$. Also, the edge label of ww_n is $l+m+n+3$. Therefore, the induced edge labels of G are distinct. Hence the graph G is a skolem mean graph.

Case 3 Let $l \le m < n$ where $n = l+m+1$ for integers $l, m \ge 1$. In this case, the required vertex labeling $f : V(G) \to \{1, 2, 3, \cdots, l+m+n+3\}$ is defined as follows:

$$\begin{aligned}
&f(u) = 1;\ f(v) = 2;\ f(w) = l+m+n+3;\\
&f(u_i) = 2i+1 \text{ for } 1 \le i \le l;\\
&f(v_j) = 2l+2j+1 \text{ for } 1 \le j \le m;\\
&f(w_k) = 2k+2 \text{ for } 1 \le k \le n-1 \text{ and}\\
&f(w_n) = l+m+n+2.
\end{aligned}$$

The corresponding edge labels are as follows:

The edge labels of uu_i is $i+1$ for $1 \le i \le l$; vv_j is $l+j+2$ for $1 \le j \le m$ and ww_k is $\dfrac{2k+l+m+n+5}{2}$ for $1 \le k \le n-1$. Also, the edge label of ww_n is $l+m+n+3$. Therefore, the induced edge labels of G are distinct. Therefore, G is a skolem mean graph.

Case 4 Let $l \le m < n$ where $n = l+m$ for integers $l, m \ge 1$. We prove such graph G is a skolem mean graph. In this case, the required vertex labeling $f : V(G) \to \{1, 2, 3, \cdots, l+m+n+3\}$ is defined as follows:

$$\begin{aligned}
&f(u) = 1;\ f(v) = 3;\ f(w) = l+m+n+3;\\
&f(u_i) = 2i \text{ for } 1 \le i \le l;\\
&f(v_j) = 2l+2j \text{ for } 1 \le j \le m;\\
&f(w_k) = 2k+3 \text{ for } 1 \le k \le n-1 \text{ and}\\
&f(w_n) = l+m+n+2.
\end{aligned}$$

Calculation shows the corresponding edge labels are as follows:

The edge labels of uu_i is $i+1$ for $1 \le i \le l$; vv_j is $l+j+2$ for $1 \le j \le m$ and ww_k is $\dfrac{2k+l+m+n+6}{2}$ for $1 \le k \le n-1$. Also, the edge label of ww_n is $l+m+n+3$. Therefore, the induced edge labels of G are distinct and G is a skolem mean graph.

Combining these discussions of Cases $1-4$, we know that G is a skolem mean graph. □

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[1] V.Balaji, D.S.T.Ramesh and A. Subramanian, Skolem mean labeling, *Bulletin of Pure and Applied Sciences*, 26E(2)(2007), 245-248.

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[4] F.Harary, *Graph Theory*, Addison-Wesley, Reading Mass, 1972.

Experience is not interesting till it begins to repeat itself, in fact, till it does that, it hardly is experience.

By Elizabeth Bowen, a British novelist.

Author Information

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Books

[4]Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest Press, 2009.
[12]W.S. Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Combinatorial speculation and combinatorial conjecture for mathematics, *International J.Math. Combin.*, Vol.1, 1-19(2007).
[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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